行星物理学（10 分）

本题由两个与行星内部相关的独立问题组成。行星表面的曲率效应可以忽略。你可能用到以下公式

$$(1+x)^{\varepsilon} \approx 1+\varepsilon x, \text{ 当 } |x| \ll 1. \tag{1}$$

A 部分. 海洋中脊（5.0 分）

考虑一个大型水容器，位于具有自由落体加速度 g 的均匀重力场中. 两个相互平行的长方形板竖直插入容器中并与容器壁无间隙地紧密接触，使得每块板浸入水中部分的长度为 h（图 1）。每块板沿 y-轴方向的宽度为 w，水密度为 ρ_0。

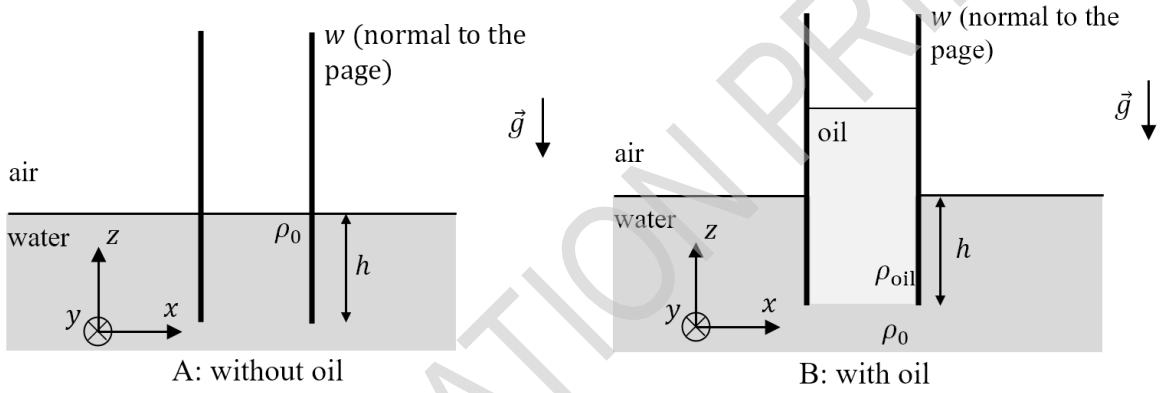


图 1. 水中的平行板（normal to the page: 垂直于纸面）

将密度为 ρ_{oil} $(\rho_{\text{oil}} < \rho_0)$ 的油倒入两板之间的空隙中，直到油的最下方达到板的下边缘为止（即两板之间的空隙中没有水了）。假设两块板和容器壁都足够高，以致油不会溢出它们。可以忽略流体的表面张力和流体间的混合。

A.1	作用在右板上的合力的 x-分量 F_x 是多少（大小与方向）?	0.8pt

图 2 显示了海洋中脊的一个横截面。它由层状结构的地幔、地壳和海水组成。地幔是由岩石构成的，在地质时间尺度内它们可以流动，因此可以当做流体处理。地壳的厚度远小于 x-方向的特征长度，因此其行为可视为能自由弯曲的薄板。在高精度下，这样的山脊可以建模为二维系统，即沿 y-轴方向没有任何变量发生变化，其中图 2 中的 y-轴垂直于纸面并指向纸里。假设沿 y-轴方向的脊长 L 比本题中其他任何长度都大得多。

在脊的中心（O 点设为坐标原点），地壳的厚度设为零。随着水平距离 x 从中心 O 开始增加，地壳变的越来越厚，并当 $x \to \infty$ 时，地壳的厚度趋于常数 D。相应地，海底从脊的顶部 O 逐渐下降到垂直深度 h（按图 2 设定的坐标系）。可以假设海水的密度 ρ_0 和温度 T_0 在空间和时上都是恒定的。对地幔也一样，其密度 ρ_1 和温度 T_1 都是恒定的。地壳的温度 T 依赖于位置，但与时间无关。

已知在高精度下，地壳物质随温度 T 线性膨胀。由于假设了海水温度和地幔温度是恒定的，所以使用重新标度的热膨胀系数是很方便的。这样有 $l(T) = l_1\left[1 - k_l(T_1 - T)/(T_1 - T_0)\right]$，其中 l 是一块地壳物质的长度，l_1 是该地壳物质在温度 T_1 时的长度，k_l 是重新标度的热膨胀系数，可以假设为常数。

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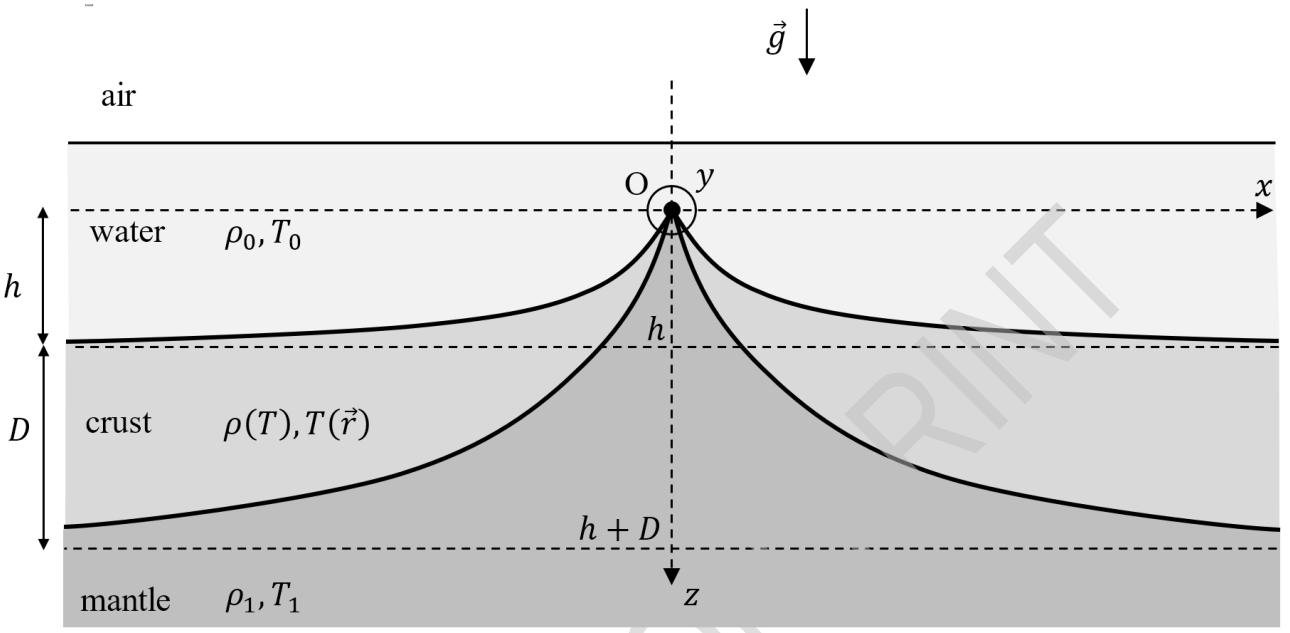


图 2. 海洋中脊. 注意图中 z-轴朝下（crust: 地壳，mantle: 地幔）

A.2 假设地壳各向同性，求出地壳密度 ρ 与温度 T 的依赖关系。假设 $|k_l| \ll 1$，把结果写成近似的形式 0.6pt

$$\rho(T) \approx \rho_1 \left[1 + k\frac{T_1 - T}{T_1 - T_0}\right], \tag{2}$$

其中忽略了 2 阶项 k_l^2 和更高阶项。由此确定常数 k 与 k_l 的关系。

已知 $k > 0$。此外可以假设地壳的热导率 κ 是恒定的。所以远离脊轴（z 轴）的地壳的温度线性依赖于深度。

A.3 假设地幔和海水在流体静力学平衡下都表现为不可压缩流体，用 h，ρ_0，ρ_1 和 k 表示出远距离处的地壳厚度 D. 忽略地幔的任何运动。 1.1pt

A.4 求出作用在地壳右半边（$x > 0$）上在水平方向上的合力 F，精确到 k 的主要阶项，并用 ρ_0，ρ_1，h, L, k 和 g 表达出来。 1.6pt

假设地壳与地球其他部分热隔绝。由于热传导，地壳的上下两个表面的温度将会越来越接近，直到地壳达到热平衡。地壳的比热是 c，且可假定为常数。

A.5 通过量纲分析或者量级分析（估算），估计特征时间 τ，它是用于刻画在远离脊轴处地壳上下表面之间的温度差减少程度的特征时间。你可以假设 τ 不依赖于地壳两个表面的初始温度。 0.9pt

B 部分. 分层介质中的地震波（5.0 分）

假设在某个行星的表面发生了一次短暂的地震。假设地震波源自位于 $z = x = 0$ 处的线源，其中 x 是水平坐标和 z 是表面以下的深度（图 3）。假设地震波的线源长度比本问题中的任何其他的长度都长得多。

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作为地震的结果，一个均匀通量的纵波 P 波，可以在 z>0 的 x-z 半平面内向各个方向发射。由于固体中的波动理论通常很复杂，在本问题中我们忽略了地震发出的所有其他波。这个行星的地壳具有分层结构，因此 P 波速度 v 依赖于深度 z，即 $v = v_0(1 + z/z_0)$，其中 v_0 是 P 波在表面处的速度，z_0 是已知的正的常数。

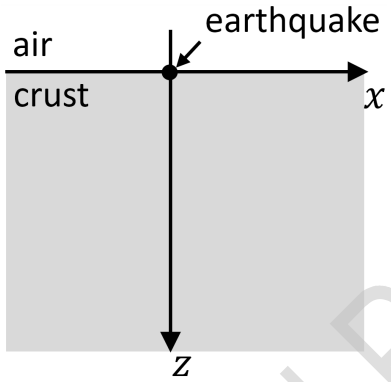


图 3. B 部分使用的坐标系（earthquake: 地震）

B.1	地震发出的一波束（a single ray）在 x-z 平面内传播，它与 z 轴的初始角度为 $0 < \theta_0 < \pi/2$。若在行星表面位于横坐标 $x_1(\theta_0) \neq 0$ 处能检测到该波束，那么该处的横坐标是什么？已知波束（the ray）的路径是一段圆弧。以 $x_1(\theta_0) = A\cot(b\theta_0)$ 的形式写出你的答案，其中 A 和 b 是待定常数。	1.5pt

如果无法确定 A 和 b，在下面问题中你可以将 $x_1(\theta_0) = A\cot(b\theta_0)$ 作为已知结果使用。假设在地震中，单位长度的震源以 P 波的形式释放到地壳的总能量为 E. 假设波从下方到达行星表面时被完全吸收。

B.2	求出行星表面所吸收的单位面积的能量密度 $\varepsilon(x)$ 如何依赖于沿表面的距离 x. 画出 $\varepsilon(x)$ 的草图，并标出已知值。	1.5pt

从现在开始，假设波在到达行星表面时被完全反射。想象一个位于 $z = x = 0$ 的设备，它与之前考虑的地震震源具有相同的几何形状。该设备能够以任意选择的角分布发射 P 波。我们让此设备发射一个具有窄角度范围的信号。特别是使信号与垂直方向的初始角度处于区间 $[\theta_0 - \frac{1}{2}\delta\theta_0, \theta_0 + \frac{1}{2}\delta\theta_0]$，其中 $0 < \theta_0 < \pi/2$, $\delta\theta_0 \ll 1$ 和 $\delta\theta_0 \ll \theta_0$。

B.3	沿着星球表面，距震源的最大距离 $x_{\max}$ 为多少时，信号达不到表面？用 θ_0、$\delta\theta_0$ 和上面给出的其他常数写出你的答案。	2.0pt

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静电透镜（10 分）

考虑一个半径为 R、均匀带电的金属环, 总电荷为 q. 环是由空心管做成的，厚度 $2a \ll R$. 这个厚度在 A、B、C 和 E 部分中，可以忽略。环平面为 xy 平面，z-轴垂直于环平面，如图 1 所示。在 A 和 B 部分中，你可能需要使用下面的公式（泰勒展开）

$$(1+x)^{\varepsilon} \approx 1 + \varepsilon x + \frac{1}{2}\varepsilon(\varepsilon-1)x^2, \text{ 当 } |x| \ll 1.$$

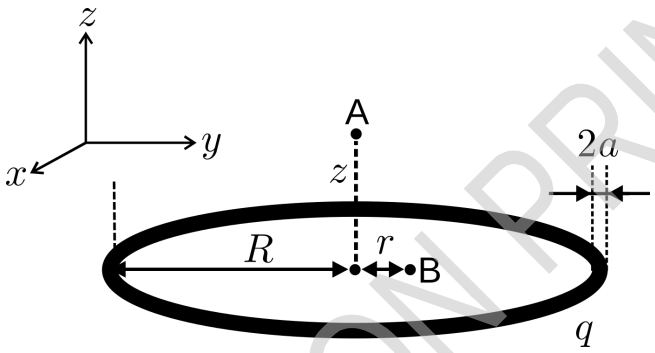


图 1. 半径为 R 的带电环

A 部分. 圆环轴上的静电势（1.0 分）

A.1	计算环的轴线上、离圆环中心距离为 z 处（图 1 中的 A 点）的静电势 $\Phi(z)$.	0.3pt
A.2	假设 $z \ll R$, 将静电势 $\Phi(z)$ 表达式展开至 z 的最低非零幂指数.	0.4pt
A.3	一个电子（质量 m，电荷 $-e$) 被置于图中 A 点 (图 1，$z \ll R$)。作用于电子上的力是多少？观察该力的表达式，确定 q 的正负使得电子的运动是一个简谐振动。运动中的电子不影响环上的电荷分布。	0.2pt
A.4	该简谐振动的角频率 ω 是多少?	0.1pt

B 部分. 圆环平面上的静电势 (1.7 分)

在本部分，你要分析圆环平面内 ($z = 0$) 当 $r \ll R$ 时 (图 1 中的 B 点）的静电势 $\Phi(r)$. 将 $\Phi(r)$ 表达式展开至 r 的最低的非零幂指数形式 $\Phi(r) \approx q(\alpha + \beta r^2)$.

B.1	求 β 的表达式，可能用到上面给出的泰勒展开。	1.5pt
B.2	一个电子被放在 B 处（图 1，$r \ll R$)。作用于电子上的力是多少? 观察该力的表达式，确定电荷 q 的正负，使得其运动为简谐振动。运动中的电子不影响环上的电荷分布。	0.2pt

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C 部分. 理想静电透镜的焦距：瞬时充电 (2.3 分)

有人打算制造一种装置来聚焦电子，也就是静电透镜。考虑以下的结构：圆环垂直 z-轴放置，如图 2 所示。我们有一个电子源，可以产生需要的非相对论电子束。电子的动能是 $E = mv^2/2$(v 是速度)，离开电子源的时刻可精确控制。系统通过编程控制，使圆环在大多数时间呈电中性，但当电子离圆环平面的距离小于 $d/2$ $(d \ll R)$ 时 (图 2 中的阴影区域，称为“活动区”(active region))，圆环充电至电量 q. 在 C 部分中，我们假设充电和放电过程是瞬时的而且电场瞬时充满空间。我们也忽略磁场的影响，并假设电子沿 z-方向的速度是恒定的。运动中的电子不影响环上的电荷分布。

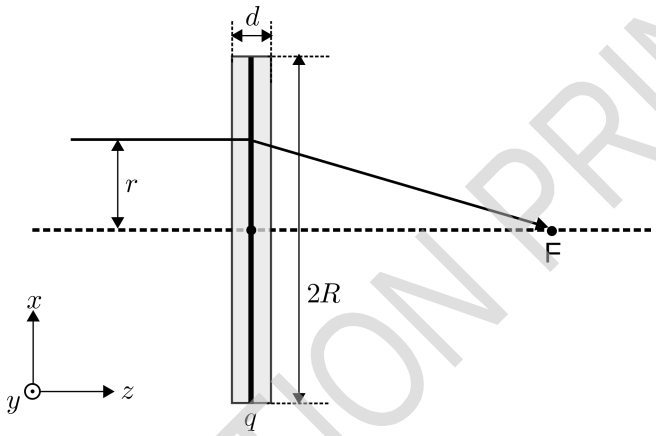


图 2. 静电透镜的模型

C.1	确定透镜焦距 f. 假设 $f \gg d$. 用 B.1 题的常数 β 和其它已知量表示。假设电子束在到达“活动区”前平行于 z-轴，且 $r \ll R$. 应该选择电荷 q 的正负使得透镜聚焦。	1.3pt

实际上，电子源放在 z-轴上，距圆环中心的距离 $b > f$. 考虑电子在到达“活动区”前不再平行于 z-轴，而是从点源相对于 z-轴以不同的角度 γ 发射，角度 γ 的范围满足 $\gamma \ll 1$ 弧度。电子将会聚焦于一点，该点离圆环中心的距离为 c.

C.2	求 c. 用问题 B.1 题的常数 β 和其它已知量表示。	0.8pt

C.3	静电透镜满足以下光学薄透镜方程吗？ $$\frac{1}{b}+\frac{1}{c}=\frac{1}{f}$$ 通过明确地计算 $1/b + 1/c$ 来回答。	0.2pt

D 部分. 环形电容器（3.0 分）

上面考虑的模型是理想化的，我们假设圆环是瞬间带电的。实际上，充电并不是瞬时，因为圆环是一个电容器，有一个有限的电容 C. 在本部分我们将分析这种电容器的特性。你可能需要以下积分：

$$\int \frac{\mathrm{d}x}{\sin x} = -\ln\left|\frac{\cos x + 1}{\sin x}\right| + \text{const}$$

和

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$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{1+x^2}\right| + \text{const.}$$

D.1	计算圆环的电容 C。考虑圆环有一个有限宽度 $2a$，但记住 $a \ll R$.	2.0pt

当电子到达“活动区”时，圆环连接到一个恒压源 V_0 上（图 3)。当电子通过“活动区”后，将圆环接地。接触电阻为 R_0，圆环自身的电阻可以忽略。

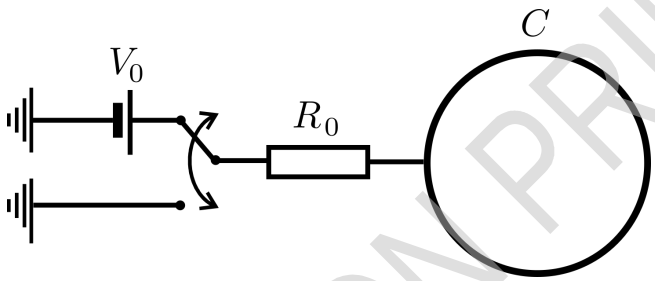


图 3. 静电透镜充电

D.2	确定环上电荷与时间的关系 $q(t)$, 并画出这种依赖关系的示意图。假设 $t=0$ 对应于电子在环平面上的时刻。当环上电荷 $q(t)$ 绝对值最大时环上电荷 q_0 是多少？圆环的电容为 C (也就是说，你没有必要用 D.1 题中电容 C 的实际表达式)。 **注意**：图 3 所示极性仅供参考。应该根据实际选择正负极性使得透镜聚焦。	1.0pt

E 部分. 近真实透镜的焦距：非瞬时充电 (2.0 分)

在本部分，我们将考虑这一近真实透镜的行为。我们将再次忽略环的宽度 $2a$，同时再次假设电子到达“活动区”前平行于 z-轴运动，然而，环的充电不再是瞬时的。

E.1	求透镜的焦距 f. 假设 $f/v \gg R_0C$, 但是 d/v 以及 R_0C 有相同的数量级。用 B 部分中的常数 β 和其它已知量表示。	1.7pt

E.2	你会发现上问中的 f 与 C 部分中求得的相似，只需将 C 部分的 q 用 q_{eff} 代替。用题中给定的量来表达 q_{eff}?	0.3pt

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粒子与波（10 分）

波粒二象性，即每个粒子都可以被描述为一个波，反之亦然，是量子力学的核心概念之一。在本题中我们将利用这一概念和其它几个基本假设，来探究微观世界中两类不同粒子—费米和玻色子—的一系列量子现象。

A 部分. 箱中的量子粒子（1.4 分）

考虑在一维势阱中运动的粒子，粒子质量为 m，其势能 $V(x)$ 表达式为:

$$V(x)=\begin{cases}0, & 0\le x\le L;\\ \infty, & x<0\,\text{or}\,x>L.\end{cases} \tag{1}$$

在该势阱中经典粒子的动能可以为任意值，但量子粒子的能级只允许取一些特定的分立的正数值。在任一这种量子态中，粒子可被描述成具有驻波形式的德布罗意波，势阱的边界为波节点（Node）。

A.1	确定势阱中量子粒子可取的最低能量值 E_{min}. 用 m, L 和普朗克常数 h 来表示。	0.4pt

粒子的最低能量态被称为基态，其它更高的能量状态都称为激发态。将所有的能量按升序排列 (能级)，用 E_n 表示，E_1 代表基态。

A.2	写出 E_n 的表达式（这里 $n=1,\,2,\,3,...$）.	0.6pt

A.3	粒子仅通过吸收或发射一个光子便可以瞬间从一个能量态跃迁到另一个态，该光子的能量等于两能级间的能量差。写出粒子从第一激发态（E_2）跃迁到基态（E_1）时所发射的光子的波长 λ_{21} 的表达式。	0.4pt

B 部分. 分子的光学性质（2.1 分）

在这一部分，我们将研究花青 Cy5 分子的几种光学特性。Cy5 分子是一种广泛使用的染料分子，其示意图见图 1a。它的光学性质主要由碳骨架决定，骨架上的碳原子之间的化学键由交替排列的单键和双键构成，见图 1b。分子两端的环和自由基 R 对光学性质的影响非常小。骨架中每个 C 原子（也包括 N 原子）中的 4 个价电子中的 3 个电子用来形成化学键，剩余的价电子被“共享”从而可在整个骨架上自由运动。每个“共享”电子感受到的净势能（net potential energy）用图 1c 中的震荡细线表示，最低点对应于 C 原子和 N 原子的位置。

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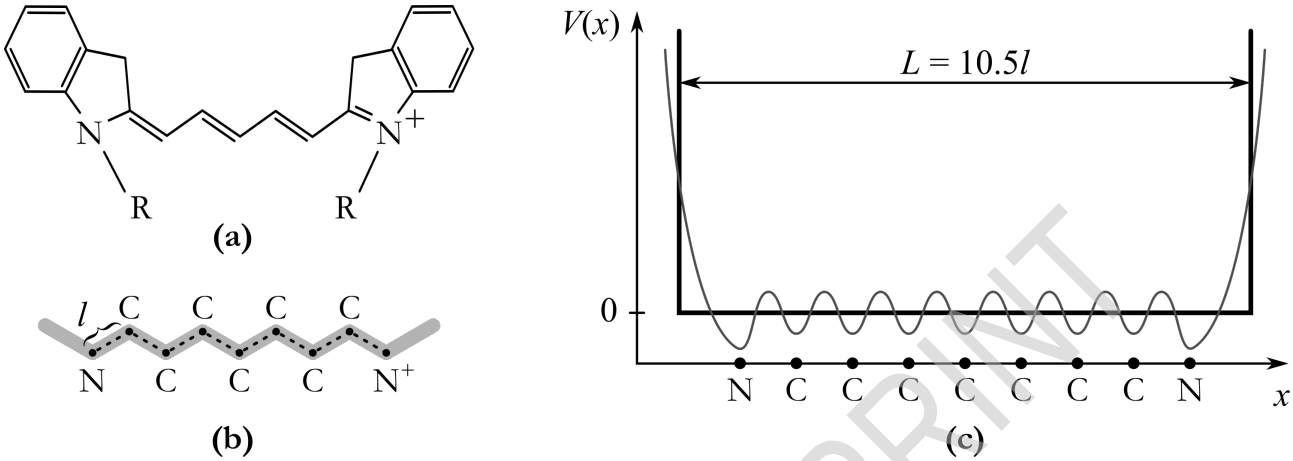


图 1. (a) 花青 Cy5 分子的化学结构（为简化起见，氢原子未画出，R 代表某种自由基）。(b)Cy5 分子的骨架，其原子间的平均距离为 l. (c) 骨架上的电子势能（细线）和势能的近似形式（粗线），该近似形式是公式 1 给出的阶跃函数

为简化起见，我们用公式.1 中的函数来近似描述电子的势能，势阱宽度为 $L = 10.5l$(见图 1c 中的粗线)，这里 $l = 140$ pm 是原子间的平均距离（参见图 1b)。总而言之，我们可将 10 个电子（7 个来自 C 原子，2 个来自 N 原子，1 个来自 N^+ 离子）看成“电子气体”，它们在 A 题中一维势阱内运动。在计算中可以忽略这些电子间的相互作用；但是，必须考虑到电子是费米子，需遵循泡利不相容原理。我们还忽略其它电子和原子核运动的影响。

B.1	计算可被 Cy5 分子吸收的光子的最长波长 λ. 假定系统中的电子开始处在基态。用 l, 物理常数和常数因子写出表达式，并计算数值结果。	0.8pt

B.2	另一种 Cy3 分子有相似的结构，但它的骨架减少了 2 个 C 原子。它的吸收光谱与 Cy5 分子相比，会红移还是蓝移？计算光谱移动 $\Delta\lambda$ 的数值。你可假定移除 2 个 C 原子不影响分子形状而只是将长度缩减两个原子间间距。	0.4pt

在激发态时，分子可通过自发辐射光子而跃迁至基态。这种辐射的平均速率 K（即：在时间 dt 内，处在激发态的分子数的相对减少为 dN/N，$K = \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t}$）由下列因素决定：辐射光子的波长 λ, 跃迁电偶极矩 d(数量级为 $d \simeq el$, 这里 e 是基本电荷)，真空介电常数 ε_0 和普朗克常数 h.

B.3	通过量纲分析，用 ε_0, h, λ 和 d 来确定自发辐射速率的表达式。表达式中的常数因子可取 $k = \frac{16}{3}\pi^3$.	0.7pt

B.4	对于 Cy5 分子，$d \approx 2.4\,el$. 估算 Cy5 分子最低激发态的平均荧光寿命，τ_{Cy5}，其为向基态辐射跃迁速率的倒数。	0.2pt

C 部分. 玻色-爱因斯坦凝聚（1.5 分）

本部分与 A，B 部分无关，我们这里将研究玻色子的集体行为。玻色子不遵循泡利不相容原理，并在低温或高密度下，经历戏剧性的玻色-爱因斯坦凝聚 (BEC) 现象。这是一种通过相变达到一种有趣的量子集体态，大量的全同粒子“凝聚”到一个量子态并表现如同一个原子。实现相变的一种典型手段是将固定数目的粒子冷却至临界温度之下。原则上，也可以通过固定温度，让数密度超过临界密度以达成相变。

我们开始探究相变时温度与数密度之间的关系。经验表明，估计他们的临界数值可通过下列的简单观测：**当对应原子方均根速率的德布罗意波长与气体中粒子间的特征距离大致相当时，BEC 就会发生。**

DELEGATION PRINT

C.1	对于无相互作用的 ^{87}Rb 气体原子在热平衡时，写出它们典型的动量 p 和德布罗意波长 λ_{dB} 的表达式，用原子质量 m, 温度 T 和物理常数表示。	0.4pt

C.2	计算气体中粒子之间的典型距离，ℓ, 与粒子数密度 n 之间的函数关系。进而推导出临界温度 T_c 表达式，用原子质量，数密度和物理常数表示。	0.5pt

为了在实验室中实现 BEC，实验者们需要将气体的温度降至 T_c= 100 nK.

C.3	若在此温度发生 BEC，Rb 气体的数密度 n_c 是多少？为方便比较，也计算普通理想气体在标准温度和气压时的数密度 n_0，即 $T_0 = 300$ K 和 $p_0 = 10^5$ Pa. 普通气体的密度 n_0 是 n_c 多少倍？你可以假定原子的质量等于 87 个原子质量单位（m_{amu}）。	0.6pt

D 部分. 三光束光学晶格 (5.0 分)

BEC 于 1995 年被首次实现。其后，实验研究向着多个方向发展。在这部分你将研究一个成果丰富的方向：将凝聚态装入由几个相干激光束干涉所产生的周期性势场中。由于干涉图样的周期性特性，也被称为光学晶格。在光学晶格中运动的原子感受到的势能 $V(\vec{r})$ 与局域光强成正比，计算中可设其表达式为：

$$V(\vec{r}) = -\alpha \langle |\vec{E}(\vec{r},t)|^2 \rangle. \tag{2}$$

其中 α 为**正的**常数；尖括号表示时间平均，取平均消除了高频振荡项。设第 *i* 束激光产生的电场为

$$\vec{E}_i = E_{0,i}\,\vec{\varepsilon}_i \cos(\vec{k}_i \cdot \vec{r} - \omega t), \tag{3}$$

其中 $E_{0,i}$ 为振幅，$\vec{k}_i$ 为波矢量，$\vec{\varepsilon}_i$ 为偏振方向的单位矢量。

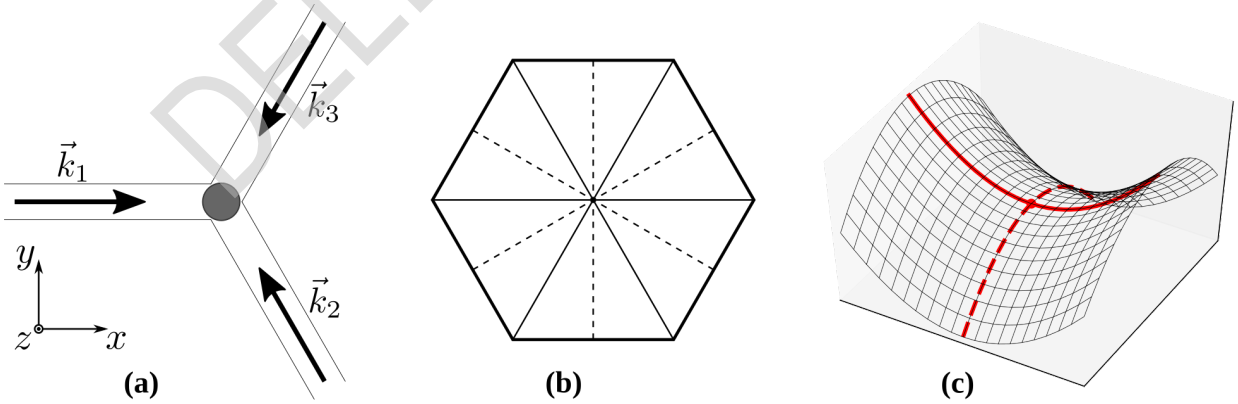


图 2. (a) 三光束光学晶格：三束波矢量为 $\vec{k}_{1,2,3}$ 的平面波在灰色圆圈所示的区域内干涉。(b) 正六边形对称性：实线和虚线表示两组对称轴。(c) 鞍点：即表面上的一个点，在该点两个正交方向的斜率都为零，但它不是所绘函数的局域极值。实线所示方向有一个表观的极小值。（如图所示）为区分真正的极小值和鞍点，还需对其垂直的方向（即虚线）进行额外分析

本部分的任务是研究三束相同强度的相干激光干涉所产生的**三角形光学晶格**。典型装置如图 2a 所示，其中三束光的偏振方向均为 z 方向，都在 xy 平面中传播，并以 120° 的相等角度相交。设 x 轴平行于波矢 $\vec{k}_1$.

DELEGATION PRINT

D.1 由 (2) 式和 (3) 式求势能 $V(\vec{r})$ 的表达式，表示为光束所在平面 $\vec{r} = (x, y)$ 的函数。 1.4pt
提示：结果可以很好地用一个常数项和三个余弦函数之和（余弦函数括号内用 $\vec{b}_i \cdot \vec{r}$ 表达）来表示。请将答案写成这种形式，并写出向量 $\vec{b}_i$ 的表达式。

D.2 所求势能具有六重旋转对称轴，即势能分布绕原点旋转 60° 的整数倍不变。对此结论提供一个简洁的论证。 0.5pt

上述对称性可简化对二维势场分布 $V(\vec{r})$ 的分析。如图 2b 所示，正六边形具有对称轴，分别连接相对的顶点（实线）和相对边的中点（虚线）。因此，在这种情况下，并不需要画出二维的势场分布图，而是通过研究 x 轴和 y 轴（它们是沿着对称轴的）从而推断出很多结论。

D.3 推导沿坐标轴的势场 $V(\vec{r})$ 的分布，即函数 $V_X(x) \equiv V(x, 0)$ 和 $V_Y(y) \equiv V(0, y)$ 的表达式。求单变量函数 $V_X(x)$ 和 $V_Y(y)$ 的极值的位置。由于这两个函数是周期性的，对于周期性重复的极小值和极大值的位置，每类仅需写出一个代表的值即可。 1.2pt

我们希望确定被称为**晶格点**的位置，即二维势能 $V(\vec{r})$ 的极小值们的位置。前面得到的单变量函数 V_X 和 V_Y 的极小点表明它们是可能的位置，但仍须检查排除掉鞍点。如图 2c 所示，当沿某一直线研究时，鞍点可能伪装成极小点而实际不是。

D.4 回顾你上题中的结果，确定光晶格中实际的极小点：定出所有最接近原点（不包含原点）的等效的极小点。相邻的极小点之间的距离 a 为多少，换言之，光晶格的**晶格常数**为多少？用激光波长 λ_{las} 表达结果。 0.8pt

超冷原子是电中性的，意味着它们之间的作用只有当两个或以上的原子占据同一个光晶格时才显著。但是，实验物理学者也探索保持原子间长程相互作用的装置。一种可能的方法是利用里德堡原子 (Rydberg atoms), 它们体型较大并有其它夸张的性质。里德堡原子是原子处在激发态，它的一个电子被激发到主量子数 n 很大的量子态。里德堡原子的大小可通过经典模型-圆轨道的半径来估算，在此轨道中电子圆周运动具有角动量 $n\hbar$, $\hbar$ 是折合普朗克常数。

D.5 计算 n 的数值，使得对应的 Rb 里德堡原子的半径与激光波长 λ_{las}= 380 nm 相当。用 λ_{las} 和物理常数表示 n，并计算其数值。 1.1pt

DELEGATION PRINT

非理想电容器（10 分）

本实验旨在研究电容器的性质。

电容器的电容（在本实验中总是指微分电容）可在与电阻 R_1 相连的情况下由电容的充电曲线 $U(t)$ 来确定。根据所用的电路，需要测量电容器的充电电流与电压的关系 $I(U)$ 并用于确定电容：

$$C(U) = \frac{\mathrm{d}q}{\mathrm{d}U} = \frac{I\mathrm{d}t}{\mathrm{d}U} = \frac{I\,(U)}{\mathrm{d}U/\mathrm{d}t}. \tag{1}$$

本实验的电路如图 1.1 所示。电路板上的开关 S1 可用于在电容器 C1 和 C2 之间切换。开关的中间位置没有任何作用，在实验中无需使用。

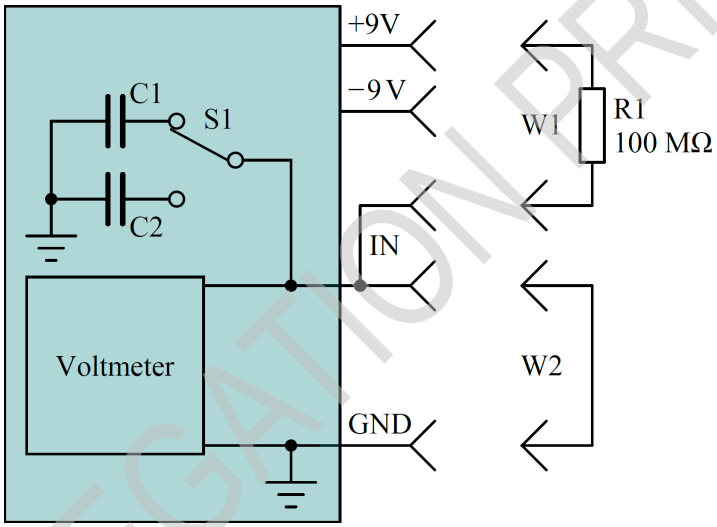


图 1.1. 实验电路图

注意：其中一个电容器中包含有电介质，其介电常数依赖于电压变化速率。为了使电压变化速率尽可能保持稳定，在测量正电压下的数据时，先让电容器充电至 9 V，然后在下降到–9 V 的过程中进行测量；而在测量负电压下的数据时，先让电容器充电至–9 V，然后在上升到 9 V 的过程中进行测量。测量的电容值会受电容器以前状态的影响，所以，在测量之前，电容器应该在起始电压至少保持 10 秒。

DELEGATION PRINT

A 部分. 室温下的电容器 (4.0 分)

测量，并画出室温下电容器 C1 和 C2 的电容与电压的关系（所有曲线画在同一张图上，用相同的坐标轴）。

A.1	测量，并在-7 V 到 7 V 的电压范围内分别画出 $C_1(U)$ 和 $C_2(U)$。在答题纸上写下 0 V，3 V 和 6 V 时的 C_1 和 C_2 值。写出用原始测量数据计算电容的公式。同时记下电路板号 (board ID) 和室温数值。	2.3pt

A.2	求电压 $U_{\text{max change}}$，在该电压时，电容器的相对电容随电压变化 $\left(\frac{\mathrm{d}C(U)}{C(U)\mathrm{d}U}\right)$ 最快。在答题纸上，写下哪个电容器（C1 或 C2）的变化最快，以及观察到最快变化时的电压 $U_{\text{max change}}$。	0.5pt

A.3	电容器 C1 和 C2 在 6 V 时的电荷 q_1 和 q_2 是多少?	1.2pt

B 部分. 校准负温度系数 (NTC) 热敏电阻（1 分）

测量在已知室温下 NTC 热敏电阻的电压（室温数值从考场的温度计上读取）。电阻随温度变化的公式（1）以及它的电路图在“实验考试-指南 G1”中。

B.1	求 NTC 热敏电阻常数 R_0。	1.0pt

C 部分 不同温度下的电容器（3 分）

C.1	分别在 40°C、65°C 和 85°C 时测量，并绘制–7 V 至 7 V 电压范围内的 $C_1(U)$ 和 $C_2(U)$。	1.3pt
C.2	分别对 0 V 和 6 V，绘制从室温到 85°C 的 $C_1(T)$ 和 $C_2(T)$ 随温度变化的曲线。	0.5pt
C.3	在答题纸上写下电容器 C1 和电容器 C2 分别在 0 V 和 6 V 时，85°C 下的电容相对于 40°C 时的电容的比值 $C(85\,^\circ\mathrm{C})/C(40\,^\circ\mathrm{C})$。	1.2pt

D 部分. 测量误差的来源（2.0 分）

在本实验之前的任务中，测量开始时需要长时间的充电。当起始充电时间较短时（0.1 至 10s）可能会有多种误差来源：

1. 漏电流。

2. 电容器中电介质的极化特性，该特性可以用依赖于过程时间尺度的介电常数来表示。

注意：隔热材料可能会吸收空气中的水分而导电。在进行漏电测量时去掉它。

确定测量 C1 和 C2 的主要误差来源。因为电容器漏电和电压表输入电流依赖于电压，所以在电压接近 9 V 时估计这些误差。为了确定误差来源并估计误差，请确定该做哪些辅助测量，并确定在什么条件下进行这些测量。回答 D.1 和 D.2 题时，可参考下表 2 个例子的格式，指出你的测量条件、测量哪些量以及根据测量得出什么结论。

注意：下表只是如何描述你的测量的例子；你需要自行确定你测量的相关条件。

应该如何书写问题 D.1 和 D.2 的答案的示例：

DELEGATION PRINT

示例 1.
本示例显示连接到测量电路的 C1 的电压变化速率在 9 V 时比在 0 V 时更快。
开关 S1 可能的位置：C1、C2
IN 接线端可能的连接：+9V、-9V、接地（GND）、空置 (Free)
初始设置：

开关 S1 位置	IN 端口连接
C1	9V

过程:

步骤	开关 S1 位置	IN 端口连接	持续时间 (s)	测量的量
1	C1	Free		$\lvert \mathrm{d}uC(t)\rvert/\mathrm{d}t$
2	C1	GND		
3	C1	Free		$\lvert \mathrm{d}uC(t)\rvert/\mathrm{d}t$

由数据可验证: $\lvert \mathrm{d}uC(t)\rvert/\mathrm{d}t\rvert_1 > \lvert \mathrm{d}uC(t)\rvert/\mathrm{d}t\rvert_3$

示例 2.
本示例显示 C1 在 9 V 时的电压变化速率大于从 0 V 开始持续 1000 秒的平均电压变化速率。
可能的开关 S1 位置：C1、C2
可能的 IN 端口连接：+9V、-9V、接地（GND）、空置 (Free)
初始设置：

开关 S1 位置	IN 端口连接
C1	9V

过程:

步骤	开关 S1 位置	IN 端口连接	持续时间 (s)	测量的量
1	C1	Free		$\lvert \mathrm{d}uC(t)\rvert/\mathrm{d}t$
2	C1	GND		
3	C1	Free		uC
4	C1	Free	1000	
5	C1	Free		uC

由数据可验证: $\lvert \mathrm{d}uC(t)\rvert/\mathrm{d}t\rvert_1 > (uC|_3 - uC|_5)/1000$

D.1 通过测量确定 $C_1(9\,\mathrm{V})$ 的主要误差来源是什么？在表格中写下测量的步骤。 1.0pt

DELEGATION PRINT

D.2	通过测量确定 $C_2(9\,\mathrm{V})$ 的主要误差来源是什么？在表格中写下测量的步骤。	1.0pt

DELEGATION PRINT

发光二极管 (LEDs)（10 分）

本实验旨在研究 LED 的电学和热学特性。测量印刷电路板 PCB 的温度时，你需要使用实验 1 的 B.1 部分得到的系数。本实验电路如图 2.1 所示。仪器指南请参照实验题 1 的说明。

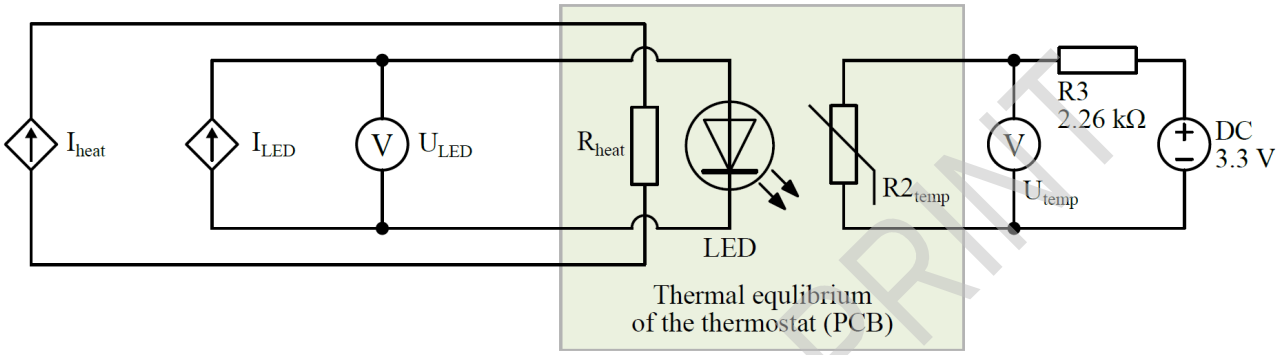


图 2.1. 实验电路。LED 由恒定电流（连续或脉冲模式）驱动，正向电压由高阻抗的电压表测量。加热和温度测量部分与实验 1 相同。印刷电路板（PCB）上的所有部件都处于热平衡。

与白炽灯由恒定电压驱动不同，LED 通常由恒定电流驱动。LED 上测得的电压与工作电流以及 LED 半导体芯片的温度有关。LED 的伏安特性的数学表达式很复杂，取决于物理和工艺参数（通常是未知的）。因此，本实验将研究 LED 电压与 LED 电流以及 LED 芯片温度 T_J 之间的关系：

$$U_\mathrm{LED} = \mathrm{function}(I_\mathrm{LED}, T_\mathrm{J}).$$

在各个电流下（I_LED），LED 半导体芯片与 PCB 之间的热阻与电功率 P 有关：

$$\frac{\Delta T}{P} = \frac{(T_\mathrm{J} - T_\mathrm{PCB})}{P}.$$

注意：LED 可被连续电流或短电流脉冲驱动。在后一种情况下，可假设脉冲持续时间足够短，从而避免 LED 的自热（例如脉冲持续时间为 1ms、测量的间隔时间为 100ms 以上）并可假设在这种驱动模式下 $T_\mathrm{J} = T_\mathrm{PCB}$。而在连续模式下 $T_\mathrm{J} > T_\mathrm{PCB}$，可以计算出热阻 $\frac{\Delta T}{P}$。

A 部分：不同温度下的伏安特性（5.0 分）

实验 2 和实验 1 中加热的物理机制是一样的。因此，你可以用实验 1 中得到的结果将热电阻的电压与其温度联系起来。或者你可以用下面的近似关系式：

$$T(U) = \frac{3500}{9.9 - \ln(\frac{1}{U} - 0.3)},$$

其中，T 是热电阻的温度，单位为开尔文。U 是热电阻的电压，单位是伏特。

测量并画出脉冲模式下从室温至 80 °C 的 LED 电流-电压关系。

A.1	测量并画出室温、40、60 和 80 °C 下 3 mA 至 50 mA 的 $I_\mathrm{LED_pulsed}(U_\mathrm{LED_pulsed}, T)$ 关系。将所有曲线画在一幅图中。	2.5pt

DELEGATION PRINT

A.2	在答题纸表格中，填写室温、40、60 和 80 °C 下，3、10、20 和 40 mA 驱动电流 $I_{\text{LED_pulsed}}$ 时的 $U_{\text{LED_pulsed}}$ 值。	1.0pt

A.3	画出主要数据点（A.2 题中所列的）的 $U_{\text{LED_pulsed}}(I_{\text{LED_pulsed}}, T)$ 关系，并计算（从图上估算）3、10、20 和 40 mA 时电压随温度的线性系数 $\Delta U(I)/\Delta T$。	1.5pt

B 部分：在连续驱动电流下测量 LED 的伏安特性（3.5 分）

B.1	加热器关闭，在连续驱动模式下测量并画出 3 mA 至 50 mA 的电流-电压关系 $I_{\text{LED_continuous}}(U_{\text{LED_continuous}})$。同时在答题纸上还要写下 3，10，20，40 mA 时的 $U_{\text{LED_continuous}}$，PCB(恒温器) 温度 T_{PCB} 以及电压差 $\Delta U = U_{\text{LED_pulsed}} - U_{\text{LED_continuous}}$ 的值。	1.5pt

B.2	由于 LED 的电阻不是常数（与电流有关），因而用到动态电阻 $\frac{\mathrm{d}U}{\mathrm{d}I}$ 来表示。利用 B.1 题的图计算 LED 的动态电阻倒数 $1/(\frac{\mathrm{d}U}{\mathrm{d}I}) = \frac{\mathrm{d}I}{\mathrm{d}U}$。在答题纸上写出 3、10、20 和 40 mA 时的 $\frac{\mathrm{d}I}{\mathrm{d}U}$ 值。在 (B.1) 图上画出这些点处的切线 $\frac{\mathrm{d}I}{\mathrm{d}U}$。	0.5pt

B.3	在连续驱动工作模式下，计算并画出半导体芯片的温度（T_{J}）与 PCB 的温度 (T_{PCB}) 的温差 $\Delta T(P)$ 作为电功率的函数（在电流为 3，10，20 和 40mA 时）。计算（从图上估算）LED 的线性热阻 $\frac{\Delta T}{P}$，并写在答题纸上。 注意：假设 LED 消耗的电能全部转化为热能，以光辐射形式消耗的能量可忽略不计。	1.5pt

C 部分：计算由温度导致的 LED 电流漂移（1.5 分）

在引言中讲到 LED 通常由恒定电流驱动，而不是恒定电压。在 B.1 题测得了 20 mA 电流时的电压，假定你决定用这个测量的恒定电压、号称 20 mA 的电流来驱动 LED。

C.1	使用 B 部分计算出的 LED 特性，计算在电压保持恒定时（即 B.1 题中测得的电压 $U(20\text{mA})$）、PCB 温度为 0 和 40 °C 下 LED 中实际流过的电流值。	1.5pt

DELEGATION PRINT

Planetary Physics (10 points)

Part A. Mid-ocean ridge (5.0 points)

A.1 (0.8 points)

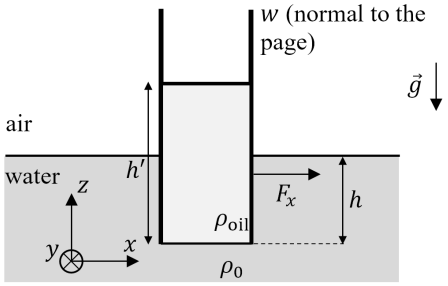


Figure 1

Let h' be the height of the column of oil (see Fig. 1). Then pressure at depth h below the water surface must be $p_h = \rho_{oil} g h = \rho_{oil} g h'$, from where $h' = \frac{\rho_0}{\rho_{oil}} h$. Horizontal force on the plate $F_x = F_1 - F_0$, where the force due to new fluid is $F_1 = \frac{\rho_{oil} g h'}{2} \cdot h' w$ and the force due to water is $F_0 = \frac{\rho_0 g h}{2} \cdot h w$.

Combining all the equation above, we get

$$F_x = \left(\frac{\rho_0}{\rho_{oil}} - 1 \right) \frac{\rho_0 g h^2 w}{2}.$$

A.1 (0.8 pt)

$$F_x = \left(\frac{\rho_0}{\rho_{oil}} - 1 \right) \frac{\rho_0 g h^2 w}{2}.$$

A.2 (0.6 points)

Consider a rectangular mass element of the crust. Since relation $l(T) = l_1 \left[1 - k_l \left(T_1 - T\right) / \left(T_1 - T_0\right)\right]$ holds for all three dimensions of the solid, its volume V satisfies

$$V = V_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0} \right)^3,$$

where V_1 is the volume at $T = T_1$. If the mass of the element is m, density is then

$$\rho(T) = \frac{m}{V} = \frac{m}{V_1} \left(1 - k_l \frac{T_1 - T}{T_1 - T_0} \right)^{-3} = \rho_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0} \right)^{-3}.$$

Since $k_l \ll 1$, this can be approximated as

$$\rho\left(T\right) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right),$$

so that $k = 3k_l$.

A.2 (0.6 pt)

$\rho\left(T\right) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right).$ $\qquad k = 3k_l.$

A.3 (1.1 points)

Since mantle behaves like a fluid in hydrostatic equilibrium, pressure $p\left(x, z\right)$ at $z = h + D$ must be the same for all x. Therefore,

$$p\left(0, h + D\right) = p\left(\infty, h + D\right).$$

Similarly, we must have

$$p\left(0, 0\right) = p\left(\infty, 0\right).$$

Hence, the change in pressure between $z = 0$ and $z = \infty$ must be the same at both $x = 0$ and $x = \infty$. At the ridge axis

$$p\left(0, h + D\right) - p\left(0, 0\right) = \rho_1 g\left(h + D\right),$$

while far away

$$p\left(\infty, h + D\right) - p\left(\infty, 0\right) = \rho_0 g h + \int_h^{h+D} \rho\left(T\left(\infty, z\right)\right) g \, \mathrm{d}z.$$

Far away from the ridge axis the two surfaces of the crust are effectively horizontal, meaning that the law of heat conduction can be written as

$$\frac{\mathrm{d}T}{\mathrm{d}z} = \text{const}.$$

Hence, after applying the relevant temperature boundary conditions,

$$T\left(\infty, z\right) = T_0 + \left(T_1 - T_0\right) \frac{z - h}{D}.$$

From all the equations above and by using the density formula given in the problem text,

$$\rho_1 g\left(h + D\right) = \rho_0 g h + \int_h^{h+D} \rho_1 \left(1 + k \frac{T_1 - T_0 - \left(T_1 - T_0\right) \frac{z-h}{D}}{T_1 - T_0}\right) g \, \mathrm{d}z,$$

from where we straightforwardly obtain

$$D = \frac{2}{k}\left(1 - \frac{\rho_0}{\rho_1}\right) h.$$

A.3 (1.1 pt)

$$D = \frac{2}{k}\left(1 - \frac{\rho_0}{\rho_1}\right)h.$$

A.4 (1.6 points)

The net horizontal force on the half of the ridge is the difference between the pressure forces acting at $x = 0$ and $x = \infty$:

$$F = L\int_0^{h+D} \left(p\left(0, z\right) - p\left(\infty, z\right)\right)\, \mathrm{d}z.$$

From considerations of the previous question, pressure at $x = 0$ is

$$p\left(0, z\right) = p\left(0, 0\right) + \rho_1 g z,$$

while very far away

$$p\left(\infty, z\right) = \begin{cases} p\left(\infty, 0\right) + \rho_0 g z & \text{if } 0 \le z \le h, \\ p\left(\infty, 0\right) + \rho_0 g h + \int_h^z \rho_1 \left(1 + k\frac{\frac{T_1 - T_0}{T_1 - T_0}\left(T_1 - T_0\right)\frac{z' - h}{D}}{T_1 - T_0}\right) g\, \mathrm{d}z' & \text{if } h \le z \le h + D. \end{cases}$$

The equations above can be combined into

$$F = L\int_0^{h+D} \left(p\left(0, 0\right) + \rho_1 g z\right)\, \mathrm{d}z - L\int_0^{h} \left(p\left(\infty, 0\right) + \rho_0 g z\right)\, \mathrm{d}z - \\ - L\int_h^{h+D} \left(p\left(\infty, 0\right) + \rho_0 g h\right)\, \mathrm{d}z - L\int_h^{h+D} \left[\int_h^z \rho_1 \left(1 + k\left(1 - \frac{z' - h}{D}\right)\right) g\, \mathrm{d}z'\right]\, \mathrm{d}z.$$

The double integral can be easily found either directly or by using a substitution $u = z - h$, $u' = z' - h$:

$$\int_h^{h+D} \left[\int_h^z \rho_1 \left(1 + k\left(1 - \frac{z' - h}{D}\right)\right) g\, \mathrm{d}z'\right]\, \mathrm{d}z = \int_0^{D} \left[\int_0^u \rho_1 \left(1 + k\left(1 - \frac{u}{D}\right)\right) g\, \mathrm{d}u'\right]\, \mathrm{d}u$$

After a straightforward integration and using $p\left(0, 0\right) = p\left(\infty, 0\right)$ as well as the result of the previous question,

$$F = gL\left[\rho_1\left(\frac{h^2}{2} + hD - \frac{kD^2}{3}\right) - \rho_0\left(\frac{h^2}{2} + hD\right)\right] = gLh^2\left(\rho_1 - \rho_0\right)\left(\frac{1}{2} + \frac{2}{3k}\left(1 - \frac{\rho_0}{\rho_1}\right)\right).$$

Since $k \ll 1$, the term with $\frac{1}{k}$ is of the leading order, hence, the required answer is

$$F \approx \frac{2gLh^2}{3k}\frac{\left(\rho_1 - \rho_0\right)^2}{\rho_1}.$$

A.4 (1.6 pt)

$$F \approx \frac{2gLh^2}{3k}\frac{(\rho_1-\rho_0)^2}{\rho_1}.$$

A.5 (0.9 points)

The timescale τ is expected to depend only on density of the crust ρ_1, its specific heat c, thermal conductivity κ and thickness D. Hence, we can write $\tau = A\rho_1^\alpha c^\beta \kappa^\gamma D^\delta$, where A is a dimensionless constant. We will obtain the powers α–δ via dimensional analysis.

Define the symbols for different dimensions: L for length, M for mass, T for time and Θ for temperature. Then τ, ρ_1, c, κ and D have dimensions T, ML^{-3}, $\mathrm{L}^2\mathrm{T}^{-2}\Theta^{-1}$, $\mathrm{MLT}^{-3}\Theta^{-1}$ and L, respectively. The resulting set of linear equations to balance the powers of length, mass, time and temperature, respectively, is

$$\begin{cases} 0 = -3\alpha + 2\beta + \gamma + \delta, \\ 0 = \alpha + \gamma, \\ 1 = -2\beta - 3\gamma, \\ 0 = -\beta - \gamma. \end{cases}$$

This gives $\alpha = \beta = 1$, $\gamma = -1$, $\delta = 2$. Hence,

$$\tau = A\frac{c\rho_1 D^2}{\kappa}.$$

A.5 (0.9 pt)

$$\tau \approx \frac{c\rho_1 D^2}{\kappa}.$$

Part B. Seismic waves in a stratified medium (5.0 points)

B.1 (1.5 points)

Seismic waves in this problem can be treated by using ray theory. Namely, their propagation is described by the Snell's law of refraction

$$n(0)\sin\theta_0 = n(z)\sin\theta,$$

where the refractive index is

$$n(z) = \frac{c}{v(z)} = \frac{c}{v_0\left(1+\frac{z}{z_0}\right)}$$

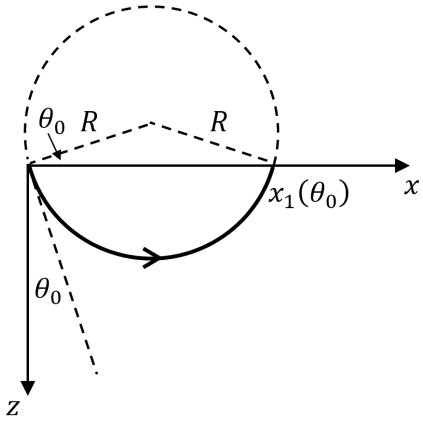


Figure 2

and c denotes the seismic wave speed in a material with refractive index $n = 1$. From the two equations above we have

$$v_0\left(1+\frac{z}{z_0}\right)\sin\theta_0 = v_0\sin\theta.$$

Method 1. Since this describes an arc of a circle, we have that at $\theta = \frac{\pi}{2}$, $z = R - R\sin\theta_0$ (fig. 2), giving

$$\left(1+\frac{R-R\sin\theta_0}{z_0}\right)\sin\theta_0 = 1,$$

from where the circle radius $R = \frac{z_0}{\sin\theta_0}$. From simple geometry we get

$$x_1(\theta_0) = 2R\cos\theta_0,$$

i.e. $A = 2z_0$ and $b = 1$.

Method 2. Implicitly differentiating $v_0\left(1+\frac{z}{z_0}\right)\sin\theta_0 = v_0\sin\theta$ gives

$$\frac{\mathrm{d}z}{z_0}\sin\theta_0 = \cos\theta\,\mathrm{d}\theta.$$

An infinitesimal ray path length $\mathrm{d}l$ is related to the change in the vertical coordinate via

$$\mathrm{d}z = \mathrm{d}l\,\cos\theta,$$

giving

$$\mathrm{d}l = \frac{z_0}{\sin\theta_0}\,\mathrm{d}\theta.$$

This is an equation of an arc of a circle of radius $R = \frac{z_0}{\sin\theta_0}$

Alternatively, instead of considering an infinitesimal ray path length $\mathrm{d}l$, one can obtain the answer by writing

$$\cot\theta = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x}.$$

The first derivative can be eliminated via Snell's law, leading to

$$\cot\theta = \frac{z_0\cos\theta}{\sin\theta_0}\frac{\mathrm{d}\theta}{\mathrm{d}x},$$

which can be integrated to get

$$x_1 = -\frac{z_0}{\sin\theta_0}\int_{\text{start}}^{\text{end}}\mathrm{d}\cos\theta = \frac{2z_0\cos\theta_0}{\sin\theta_0},$$

where we used Snell's law again to get that the ray has $\cos\theta = -\cos\theta_0$ at the point where it reaches the surface.

B.1 (1.5 pt)

$x_1(\theta_0) = 2z_0\cot\theta_0$.

B.2 (1.5 points)

In two dimensions, $\frac{E}{\pi}\,\mathrm{d}\theta_0$ is the energy carried by rays that are emitted within interval $[\theta_0, \theta_0+\mathrm{d}\theta_0)$. On the other hand, the energy carried by rays that arrive at $[x, x+\mathrm{d}x)$ is $\varepsilon\,\mathrm{d}x$. Therefore,

$$\varepsilon = \frac{E}{\pi}\left|\frac{\mathrm{d}\theta_0}{\mathrm{d}x}\right|.$$

Using the result of question B.1,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta_0} = -\frac{Ab}{\sin^2(b\theta_0)} = -Ab\left(1+\cot^2(b\theta_0)\right) = -\frac{b\left(A^2+x^2\right)}{A}.$$

Hence,

$$\varepsilon(x) = \frac{EA}{\pi b\left(A^2+x^2\right)} = \frac{2Ez_0}{\pi\left(4z_0^2+x^2\right)}.$$

This function is plotted in Fig. 3.

B.2 (1.5 pt)

$\varepsilon(x) = \dfrac{EA}{\pi b\left(A^2+x^2\right)} = \dfrac{2Ez_0}{\pi\left(4z_0^2+x^2\right)}$.

Sketch is shown in Fig. 3.

B.3 (2.0 points)

Define $x_- = x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right)$ and $x_+ = x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)$. To the leading order in $\delta\theta_0$, $x_- \approx x_+ \approx x_1(\theta_0)$. With each reflection of the signal, the horizontal distance between the points where the edges of

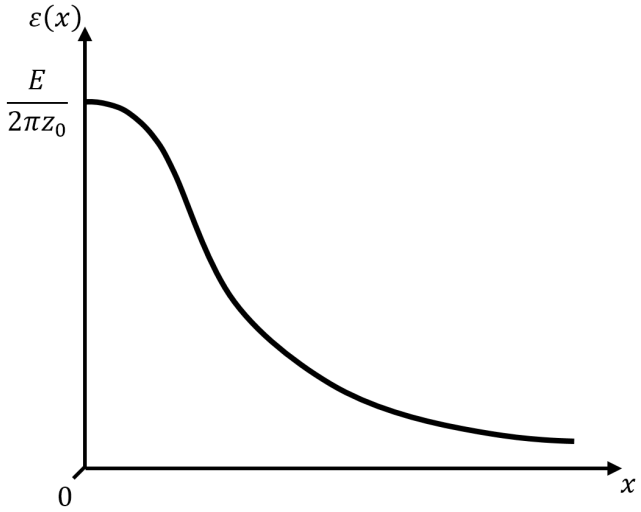


Figure 3. Plot of the function $\varepsilon(x)$.

the signal reflect increases by $|x_+ - x_-| = x_- - x_+$. When moving along the positive x-axis, these zones get wider until they overlap. If this happens after N reflections, then

$$N \approx \frac{x_1(\theta_0)}{x_- - x_+},$$

where the approximate sign tends to equality as $\delta\theta_0 \to 0$.

The position where the zones start to overlap is at $x_{\max} = Nx_1(\theta_0)$. Therefore,

$$x_{\max} = \frac{x_1(\theta_0)^2}{x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)}.$$

Since $\delta\theta_0 \ll \theta_0$, we can approximate

$$x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right) \approx -\frac{\mathrm{d}x_1(\theta_0)}{\mathrm{d}\theta_0}\delta\theta_0 = \frac{Ab}{\sin^2(b\theta_0)}\delta\theta_0.$$

Combining the last two equations and substituting the $x_1(\theta_0)$ expression gives

$$x_{\max} = \frac{Ab\cos^2(b\theta_0)}{\delta\theta_0} = \frac{2z_0\cos^2\theta_0}{\delta\theta_0}.$$

B.3 (2.0 pt)

$$x_{\max} = \frac{Ab\cos^2(b\theta_0)}{\delta\theta_0} = \frac{2z_0\cos^2\theta_0}{\delta\theta_0}.$$

Electrostatic lens (10 points)

Part A. Electrostatic potential on the axis of the ring (1 point)

A.1 (0.3 points)

The linear charge density of the ring is $\lambda = q/(2\pi R)$. All the points of the ring are situated a distance $\sqrt{R^2+z^2}$ away from point A. Integrating over the whole ring we readily obtain:

$$\Phi(z) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}}.$$

A.1 (0.3 pt)

$$\Phi(z) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}}.$$

A.2 (0.4 points)

Using an expansion in powers of z we obtain:

$$\Phi(z) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}} = \frac{q}{4\pi\varepsilon_0 R}\frac{1}{\sqrt{1+\left(\frac{z}{R}\right)^2}} \approx \frac{q}{4\pi\varepsilon_0 R}\left(1-\frac{z^2}{2R^2}\right).$$

A.2 (0.4 pt)

$$\Phi(z) \approx \frac{q}{4\pi\varepsilon_0 R}\left(1-\frac{z^2}{2R^2}\right).$$

A.3 (0.2 points)

The potential energy of the electron is $V(z) = -e\Phi(z)$. The force acting on the electron is

$$F(z) = -\frac{\mathrm{d}V(z)}{\mathrm{d}z} = +e\frac{\mathrm{d}\Phi}{\mathrm{d}z} = -\frac{qe}{4\pi\varepsilon_0 R^3}z.$$

If this is a restoring force, it should be negative for positive z. Thus, $q > 0$.

A.3 (0.2 pt)

$$F(z) = -\frac{qe}{4\pi\varepsilon_0 R^3}z. \qquad q > 0.$$

A.4 (0.1 points)

The equation of motion for an electron is

$$m\ddot{z} + \frac{qe}{4\pi\varepsilon_0 R^3} z = 0$$

(here dots denote time derivatives). We therefore get

$$\omega = \sqrt{\frac{qe}{4\pi m\varepsilon_0 R^3}}.$$

A.4 (0.1 pt)

$\omega = \sqrt{\frac{qe}{4\pi m\varepsilon_0 R^3}}$.

Part B. Electrostatic potential in the plane of the ring (1.7 points)

B.1 (1.5 points)

There are two different ways to solve this problem: (i) using direct integration; (ii) using Gauss's law and the result of part A.

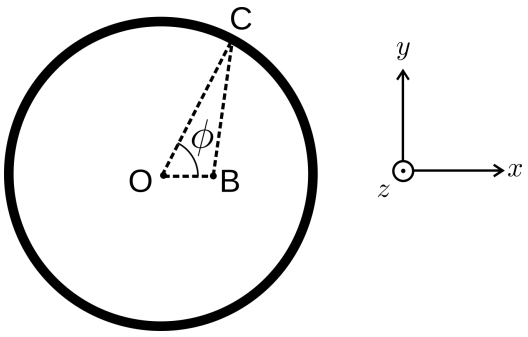


Figure 1: Calculating electrostatic potential in the plane of the ring through direct integration.

(i) **Direct integration**. We will follow the notations of Figure 1. Since the potential has cylindrical symmetry, let the point B, where we calculate the potential, be on the x-axis. Let

$$|\mathrm{OB}| = r; |\mathrm{OC}| = R.$$

Thus:

$$|\mathrm{BC}|^2 = R^2 + r^2 - 2Rr\cos\phi.$$

Electrostatic potential created by ring element $\mathrm{d}\phi$ at the point B:

$$\mathrm{d}\Phi = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\,\mathrm{d}\phi}{\sqrt{R^2+r^2-2Rr\cos\phi}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,\mathrm{d}\phi}{\sqrt{1+\frac{r^2}{R^2}-2\frac{r}{R}\cos\phi}}.$$

Using the expansion given in the formulation of the problem for $\varepsilon = -1/2$ we have:

$$\mathrm{d}\Phi \approx \frac{\lambda\,\mathrm{d}\phi}{4\pi\varepsilon_0}\left[1-\frac{1}{2}\left(\frac{r^2}{R^2}-2\frac{r}{R}\cos\phi\right)+\frac{3}{8}\left(\frac{r^2}{R^2}-2\frac{r}{R}\cos\phi\right)^2\right].$$

Ignoring the terms of the order r^3 and r^4 we get:

$$\mathrm{d}\Phi \approx \frac{\lambda\,\mathrm{d}\phi}{4\pi\varepsilon_0}\left[1+\frac{r}{R}\cos\phi+\frac{r^2}{R^2}\left(\frac{3}{2}\cos^2\phi-\frac{1}{2}\right)\right].$$

Integrating over all angles we finally obtain:

$$\Phi(r) = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{2\pi}\left[1+\frac{r}{R}\cos\phi+\frac{r^2}{R^2}\left(\frac{3}{2}\cos^2\phi-\frac{1}{2}\right)\right]\mathrm{d}\phi.$$

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0 R}\left(1+\frac{r^2}{4R^2}\right).$$

From here, comparing with the expression $\Phi(r) = q(\alpha+\beta r^2)$, we obtain

$$\beta = \frac{1}{16\pi\varepsilon_0 R^3}.$$

(ii) **Gauss's law.**

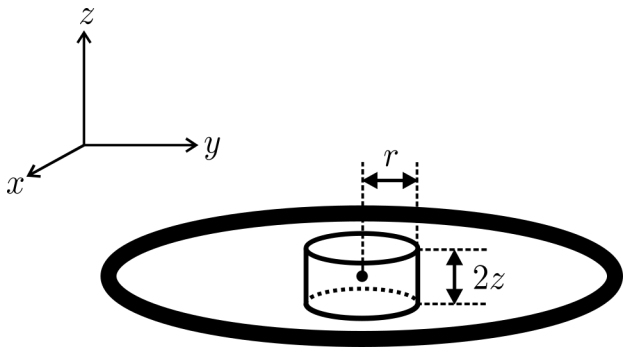


Figure 2: Calculating electrostatic potential in the plane of the ring via Gauss's law.

Let us analyze a small cylinder of radius r. The center of the cylinder coincides with the center of the ring. In part A we analyzed the potential along the z-axis, while in this part we analyze the potential along the radius r. For any $z \ll R$ and $r \ll R$ the potential has an expression:

$$\Phi(z,r) = \frac{q}{4\pi\varepsilon_0 R}\left(1-\frac{z^2}{2R^2}\right)+q\beta r^2.$$

The lowest order terms are quadratic in r and z. Due to reflection symmetry the potential does not contain terms of the type rz. This, for example, immediately gives us $\alpha = 1/(4\pi\varepsilon_0 R)$. Thus, for small r and z electric fields in the radial and axial directions are:

$$\mathcal{E}_z(z,r) = +\frac{q}{4\pi\varepsilon_0 R^3} z, \qquad \mathcal{E}_r(z,r) = -2q\beta r.$$

Applying Gauss's law to the cylinder we obtain:

$$\oint \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = 0 \qquad \Rightarrow \qquad \int\limits_{\text{side}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} + \int\limits_{\text{base}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = 0.$$

The second integral is:

$$\int\limits_{\text{base}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = 2\pi r^2 \mathcal{E}_z(z,r) = \frac{qzr^2}{2\varepsilon_0 R^3}.$$

The first integral is:

$$\int\limits_{\text{side}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = 4\pi r z \mathcal{E}_r(z,r) = -8\pi q\beta r^2 z.$$

Gauss's theorem thus gives:

$$\frac{qzr^2}{2\varepsilon_0 R^3} - 8\pi q\beta r^2 z = 0.$$

This immediately yields

$$\beta = \frac{1}{16\pi\varepsilon_0 R^3},$$

which agrees with the result obtained via direct integration.

B.1 (1.5 pt)

$\beta = \dfrac{1}{16\pi\varepsilon_0 R^3}.$

B.2 (0.2 points)

The potential of the electron is $V(r) = -e\Phi(r)$. Force acting on the electron in the xy plane is

$$F(r) = -\frac{\mathrm{d}V(r)}{\mathrm{d}r} = +e\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{qe}{8\pi\varepsilon_0 R^3} r.$$

To have oscilations we need the force to be negative for $r > 0$. Thus, $q < 0$.

B.2 (0.2 pt)

$F(r) = +\dfrac{qe}{8\pi\varepsilon_0 R^3} r. \qquad q < 0.$

Part C. The focal length of the idealized electrostatic lens (2.3 points)

C.1 (1.3 points)

Let us consider an electron with the velocity $v = \sqrt{2E/m}$ at a distance r from the "optical" axis (Figure 2 of the problem). The electron crosses the "active region" of the lens in time

$$t = \frac{d}{v}.$$

The equation of motion in the r direction:

$$m\ddot{r} = 2eq\beta r.$$

During the time the electron crosses the active region of the lens, the electron acquires radial velocity:

$$v_r = \frac{2eq\beta r}{m}\frac{d}{v} < 0.$$

The lens will be focusing if $q < 0$. The time it takes for an electron to reach the "optical" axis is:

$$t' = \frac{r}{|v_r|} = -\frac{mv}{2eq\beta d}.$$

During this time the electron travels in the z-direction a distance

$$\Delta z = t'v = -\frac{mv^2}{2eq\beta d} = -\frac{E}{eqd\beta}.$$

Δz does not depend on the radial distance r, therefore all electron will cross the "optical" axis (will be focused) in the same spot. Thus,

$$f = -\frac{E}{eqd\beta}.$$

C.1 (1.3 pt)

$f = -\frac{E}{eqd\beta}$.

C.2 (0.8 points)

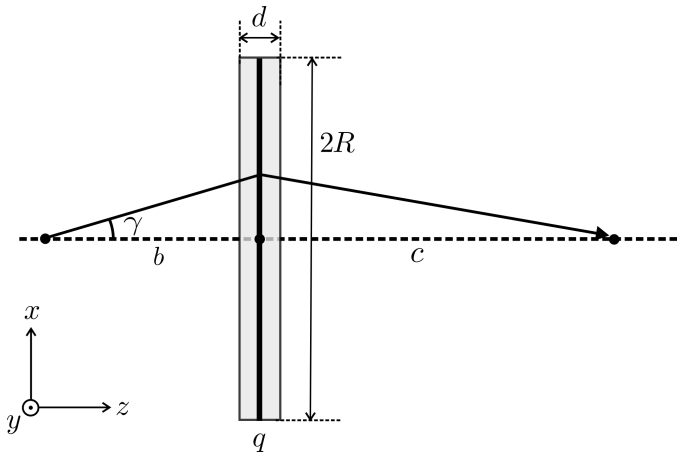


Figure 3: Focusing of electrons.

Let us consider an electron emitted an an angle γ to the optical axis (Figure 3). Its initial velocity in the radial direction is:

$$v_{r;0} = v\sin\gamma \approx v\gamma \approx v\frac{r}{b},$$

where r is the radial distance of the electron when it reaches the plane of the ring. The velocity in the z-direction is

$$v_z = v\cos\gamma \approx v.$$

For small angles γ the additional velocity in the r-direction acquired in the "active region" is the same as in part C.1. Thus, the radial velocity after crossing the active region is

$$v_r = v\frac{r}{b} + \frac{2eq\beta r}{m}\frac{d}{v},$$

where the first term is positive and the second term is negative, since $q < 0$. If the electrons are focused, then $v_r < 0$ (this can be verified after obtaining the final result). The electron will reach the optical axis in time

$$t' = \frac{r}{|v_r|} = -\frac{r}{\frac{2eq\beta r}{m}\frac{d}{v} + v\frac{r}{b}} = -\frac{1}{\frac{2eq\beta}{m}\frac{d}{v} + \frac{v}{b}}.$$

During this time it will travel a distance

$$c = t'v = -\frac{1}{\frac{2eq\beta}{m}\frac{d}{v^2} + \frac{1}{b}} = -\frac{1}{\frac{eq\beta d}{E} + \frac{1}{b}}.$$

C.2 (0.8 pt)

$$c = -\frac{1}{\frac{eq\beta d}{E} + \frac{1}{b}}.$$

C.3 (0.2 pt)

From the previous answer we obtain:

$$\frac{1}{b}+\frac{1}{c}=-\frac{eq\beta d}{E}.$$

Comparing with the answer of C.1 we immediately obtain

$$\frac{1}{b}+\frac{1}{c}=\frac{1}{f},$$

i.e. the equation of a thin optical lens is valid for an electrostatic lens as well.

C.3 (0.2 pt)

The equation of a thin optical lens $\frac{1}{b}+\frac{1}{c}=\frac{1}{f}$ is **valid** for an electrostatic lens.

Part D. The ring as a capacitor (3 points)

D.1 (2.0 points)

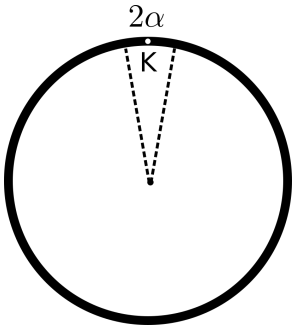


Figure 4: Calculation of the capacitance of the ring.

Let us sub-divide the entire ring into two parts: a part corresponding to the angle $2\alpha \ll 1$, and the rest of the ring, as shown in Figure 4. While the angle is small in comparison to 1, let us assume that the length of the first part, αR, is still large compared to a ($\alpha R \gg a$). Let us calculate the electrostatic potential Φ at point K. It it a sum of two terms: the first one produced by the cut-out part with an angle 2α (contribution Φ_1) and the second one originating from the rest of the ring (contribution Φ_2).

Contribution Φ_1. Since $\alpha \ll 1$, we can neglect the curvature of the cylinder that is cut out from the ring. The linear charge density on the ring is $\lambda = \frac{q}{2\pi R}$. The potential at the center of the

cylinder is then given by an integral:

$$\Phi_1 = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{2\pi R}\int_0^{\alpha R}\frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \frac{q}{4\pi^2\varepsilon_0 R}\int_0^{\alpha R}\frac{\mathrm{d}(x/a)}{\sqrt{1+(x/a)^2}} = \frac{q}{4\pi^2\varepsilon_0 R}\int_0^{\alpha R/a}\frac{\mathrm{d}y}{\sqrt{1+y^2}}.$$

Using the integral provided in the description of the problem we get:

$$\Phi_1 = \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(y+\sqrt{1+y^2}\right)\Big|_0^{\alpha R/a} = \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{\alpha R}{a}+\sqrt{1+\left(\frac{\alpha R}{a}\right)^2}\right).$$

As $\alpha R \gg a$,

$$\Phi_1 \approx \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{2\alpha R}{a}\right).$$

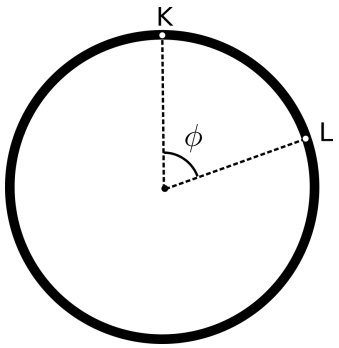


Figure 5: Calculation of the capacitance of the ring

Contribution Φ_2. In this case we can neglect the thickness a. Using the cosine theorem we can derive the distance between points K and L of Figure 5:

$$|\mathrm{KL}| = 2R\sin\frac{\phi}{2}.$$

The contribution Φ_2 can then be written as an integral:

$$\Phi_2 = 2\frac{q}{2\pi}\frac{1}{4\pi\varepsilon_0}\int_\alpha^\pi\frac{\mathrm{d}\phi}{2R\sin\frac{\phi}{2}} = \frac{q}{8\pi^2\varepsilon_0 R}\int_\alpha^\pi\frac{\mathrm{d}\phi}{\sin\frac{\phi}{2}} = \frac{q}{4\pi^2\varepsilon_0 R}\int_\alpha^\pi\frac{\mathrm{d}\left(\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}} = \frac{q}{4\pi^2\varepsilon_0 R}\int_{\alpha/2}^{\pi/2}\frac{\mathrm{d}\chi}{\sin\chi}.$$

Using the integral from the formulation of the problem, we calculate:

$$\int_{\alpha/2}^{\pi/2}\frac{\mathrm{d}\chi}{\sin\chi} = -\ln\left(\frac{\cos\chi+1}{\sin\chi}\right)\Big|_{\alpha/2}^{\pi/2} = \ln\left(\frac{\cos\alpha/2+1}{\sin\alpha/2}\right) \approx \ln\left(\frac{4}{\alpha}\right)$$

for $\alpha \ll 1$. Therefore

$$\Phi_2 \approx \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{4}{\alpha}\right).$$

The total potential and capacitance. The total potential is the sum of Φ_1 and Φ_2:

$$\Phi = \Phi_1 + \Phi_2 = \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{2\alpha R}{a}\right) + \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{4}{\alpha}\right) = \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{8R}{a}\right).$$

α drops out from the expression. From here we obtain the capacitance $C = q/\Phi$:

$$C = \frac{4\pi^2\varepsilon_0 R}{\ln\left(\frac{8R}{a}\right)}.$$

$C \to 0$ as $a \to 0$.

D.1 (2.0 pt)

$$C = \frac{4\pi^2\varepsilon_0 R}{\ln\left(\frac{8R}{a}\right)}.$$

D.2 (1.0 point)

Let $q(t)$ be the charge on the ring at a time t. Potential of the disk is thus $q(t)/C$. Voltage drop of the resistor is $R_0 I(t) = R_0\,\mathrm{d}q/\mathrm{d}t$. Therefore for time $-\frac{d}{2v} < t < \frac{d}{2v}$:

$$\frac{q(t)}{C} + R_0\frac{\mathrm{d}q}{\mathrm{d}t} = V_0.$$

Integrating this equation and keeping in mind that $q(t) = 0$ at $t = -d/(2v)$, we get:

$$q(t) = CV_0\left(1 - \mathrm{e}^{-\frac{d}{2vR_0C}}\mathrm{e}^{-\frac{t}{R_0C}}\right).$$

The charge attains the largest absolute value at $t = d/(2v)$. The value of the charge at this time is:

$$q_0 = CV_0\left(1 - \mathrm{e}^{-\frac{d}{vR_0C}}\right).$$

When $t > \frac{d}{2v}$, we get:

$$\frac{q(t)}{C} + R_0\frac{\mathrm{d}q}{\mathrm{d}t} = 0.$$

From here:

$$q(t) = q_0\mathrm{e}^{-\frac{t}{R_0C}+\frac{d}{2vR_0C}} = CV_0\left(\mathrm{e}^{\frac{d}{2vR_0C}} - \mathrm{e}^{-\frac{d}{2vR_0C}}\right)\mathrm{e}^{-\frac{t}{RC}}.$$

Therefore, we obtain:

$$q(t) = \begin{cases} 0 & \text{for } t < -\frac{d}{2v}; \\ CV_0\left(1 - \mathrm{e}^{-\frac{d}{2vR_0C}}\mathrm{e}^{-\frac{t}{R_0C}}\right) & \text{for } -\frac{d}{2v} < t < \frac{d}{2v}; \\ CV_0\left(\mathrm{e}^{\frac{d}{2vR_0C}} - \mathrm{e}^{-\frac{d}{2vR_0C}}\right)\mathrm{e}^{-\frac{t}{R_0C}} & \text{for } t > \frac{d}{2v}. \end{cases}$$

For a lens to be focusing we require that charge is negative, therefore $V_0 < 0$. The dependence of charge on time is shown in Figure 6.

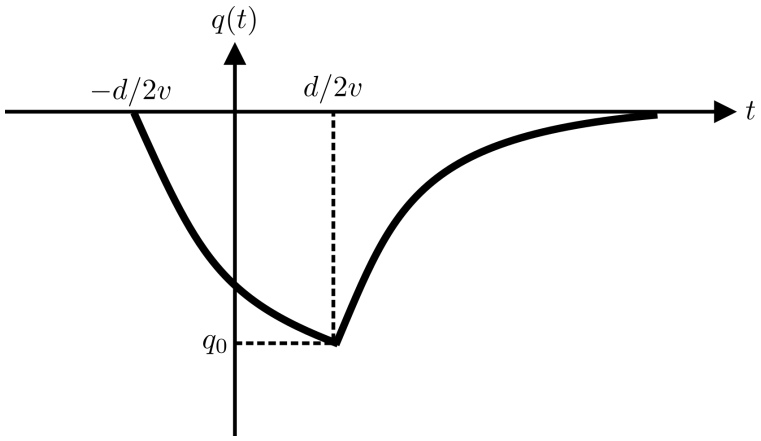


Figure 6: Charge on the ring as a function of time.

D.2 (1.0 pt)

For $-\dfrac{d}{2v} < t < \dfrac{d}{2v}$, $\quad q\left(t\right) = CV_0\left(1 - \mathrm{e}^{-\frac{d}{2vR_0C}}\mathrm{e}^{-\frac{t}{R_0C}}\right)$.

For $t > \dfrac{d}{2v}$, $\quad q\left(t\right) = CV_0\left(\mathrm{e}^{\frac{d}{2vR_0C}} - \mathrm{e}^{-\frac{d}{2vR_0C}}\right)\mathrm{e}^{-\frac{t}{R_0C}}$.

$q_0 = CV_0\left(1 - \mathrm{e}^{-\frac{d}{vR_0C}}\right)$. Schematic plot of this function is shown in Figure 6.

Part E. Focal length of a more realistic lens (2 points)

E.1 (1.7 points)

Like in part C, the radial equation of motion of an electron is:

$$m\ddot{r} = 2eq(t)\beta r,$$

where in this case $q(t)$ depends on time. Using the notation $\eta = 2e\beta/m$, we obtain:

$$\ddot{r} - \eta q(t) r = 0.$$

As $f/v \gg R_0C$, then during charging–decharging the electron does not substantially change its radial position r, and we can assume r to be constant during the entire charging–decharging process. In this case the acquired vertical velocity is

$$v_r = \eta r \int_{-d/(2v)}^{\infty} q(t)\,\mathrm{d}t.$$

We can use the derived equations for $q(t)$ and find the integrals. The integral $\int_{-d/(2v)}^{d/(2v)} q(t)\,\mathrm{d}t$ is (using the notation $d/v = t_0$, $R_0 C = \tau$, $CV_0 = Q_0$):

$$\int_{-t_0/2}^{t_0/2} q(t)\,\mathrm{d}t = \int_{-t_0/2}^{t_0/2} Q_0 \left(1 - \mathrm{e}^{-\frac{t_0}{2\tau}} \mathrm{e}^{-\frac{t}{\tau}}\right) \mathrm{d}t = Q_0 \left(t_0 - \tau \left[1 - \mathrm{e}^{-t_0/\tau}\right]\right).$$

The integral $\int_{d/(2v)}^{\infty} q(t)\,\mathrm{d}t$ is

$$\int_{t_0/2}^{\infty} Q_0 \left(e^{\frac{t_0}{2\tau}} - e^{-\frac{t_0}{2\tau}}\right) \mathrm{e}^{-\frac{t}{\tau}}\,\mathrm{d}t = Q_0 \tau \left[1 - \mathrm{e}^{-t_0/\tau}\right].$$

Adding the two integrals we obtain for the final integral:

$$\int_{-t_0/2}^{\infty} q(t)dt = Q_0 t_0.$$

Interestingly, it does not depend on $\tau = R_0 C$. Therefore, the acquired vertical velocity of the electron is

$$v_r = \eta r \frac{CV_0 d}{v} = \frac{2e\beta CV_0 dr}{mv}.$$

Following the logic similar to part C, we derive the focal length

$$f = -\frac{E}{eCV_0 d\beta}.$$

E.1 (1.7 pt)

$f = -\dfrac{E}{eCV_0 d\beta}.$

E.2 (0.3 points).

Comparing $f = -E/(eCV_0 d\beta)$ with $f = -E/(eqd\beta)$ from part C we immediataly obtain $q_{\text{eff}} = CV_0$.

E.2 (0.3 pt)

$q_{\text{eff}} = CV_0.$

Particles and Waves (10 points)

Part A. Quantum particle in a box (1.4 points)

A.1 (0.4 points)

The width of the potential well (L) should be equal to the half of the wavelength of the de Broglie standing wave $\lambda_{\text{dB}} = h/p$, here h is the Planck's constant and p is the momentum of the particle. Thus $p = h/\lambda_{\text{dB}} = h/(2L)$, and the minimal possible energy of the particle is

$$E_{\text{min}} = \frac{p^2}{2m} = \frac{h^2}{8mL^2}.$$

A.1 (0.4 pt)

$E_{\text{min}} = \dfrac{h^2}{8mL^2}.$

A.2 (0.6 points)

The potential well should fit an integer number of the de Broglie half-wavelengths: $L = \frac{1}{2}\lambda_{\text{dB}}^{(n)} \cdot n$, $n = 1, 2, \ldots$. Therefore, particle's momentum, corresponding to the de Broglie wavelength $\lambda_{\text{dB}}^{(n)}$ is

$$p_n = \frac{h}{\lambda_{\text{dB}}^{(n)}} = \frac{hn}{2L},$$

and the corresponding energy is

$$E_n = \frac{p_n^2}{2m} = \frac{h^2 n^2}{8mL^2}, \qquad n = 1, 2, 3, \ldots. \tag{1}$$

A.2 (0.6 pt)

$E_n = \dfrac{h^2 n^2}{8mL^2} = E_{\text{min}} n^2.$

A.3 (0.4 points)

The energy of the emitted photon, $E = hc/\lambda$ (here c is the speed of light and λ is the photon's wavelength) should be equal to the energy difference $\Delta E = E_2 - E_1$, therefore

$$\lambda_{21} = \frac{hc}{E_2 - E_1} = \frac{8mcL^2}{3h}.$$

A.3 (0.4 pt)

$\lambda_{21} = \dfrac{8mcL^2}{3h}$.

Part B. Optical properties of molecules (2.1 points)

B.1 (0.8 points)

Taking into account the Pauli exclusion principle, each energy level E_n is occupied by two electrons with spins oriented in the opposite directions. As a results, 10 electrons fill the lowest 5 states, and the absorption of the photon of the longest wavelength corresponds to the transition of one electron from the occupied E_5 to the unoccupied E_6 energy state:

$$\frac{hc}{\lambda} = E_6 - E_5,$$

where E_6 and E_5 can be found from Eq. 1, where m is replaced with the electron mass m_e. Hence we obtain:

$$\lambda = \frac{c \cdot 8m_\mathrm{e}L^2}{h\,(6^2 - 5^2)} = \frac{10.5^2 \cdot 8}{11}\frac{m_\mathrm{e}cl^2}{h} = \frac{882}{11}\frac{m_\mathrm{e}cl^2}{h} \approx 647\,\mathrm{nm}.$$

This result correspond precisely to the experimental value the peak position of the Cy5 absorption spectrum.

B.1 (0.8 pt)

Expression: $\lambda = \dfrac{882}{11}\dfrac{m_\mathrm{e}cl^2}{h}$. Numerical value: $\lambda = 647\,\mathrm{nm}$.

B.2 (0.4 points)

In the similar model for the Cy3 molecule, there are 8 electrons in the box of length $L = 8.5l$, thus photon's absorption corresponds to the $E_4 \to E_5$ transition. Taking into account the result of question B1, we obtain

$$\lambda_{\mathrm{Cy3}} = \frac{8.5^2 \cdot 8}{(5^2 - 4^2)}\frac{m_\mathrm{e}cl^2}{h} \approx 518\,\mathrm{nm},$$

i. e. the absorption spectrum of the Cy3 molecule is shifted by $\Delta\lambda \approx 129\,\mathrm{nm}$ to the blue comparing to that of the Cy5 molecule. The experimental value is $\lambda_{\mathrm{Cy3}}^{(\mathrm{exp})} = 548\,\mathrm{nm}$, so that our model catches general properties of these dye molecules rather well.

B.2 (0.4 pt)

Absorption spectrum of Cy3 is shifted to the (check): ☒ **bluer** ☐ redder

spectral region by $\Delta\lambda \approx 129\,\mathrm{nm}$.

B.3 (0.7 points)

Let us assume

$$K = k\varepsilon_0^{\alpha} h^{\beta} \lambda^{\gamma} d^{\delta}. \tag{2}$$

The SI units of the relevant quantities are:

$$[\varepsilon_0] = \frac{\mathrm{A}^2 \cdot \mathrm{s}^4}{\mathrm{kg} \cdot \mathrm{m}^3}, \qquad [h] = \frac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}}, \qquad [\lambda] = \mathrm{m}, \qquad [d] = \mathrm{A} \cdot \mathrm{s} \cdot \mathrm{m}, \qquad [K] = \mathrm{s}^{-1}.$$

By plugging these expressions into Eq. 2 we obtain a simple system of linear equations for the unknown powers α, β, γ, and δ:

$$2\alpha + \delta = 0, \qquad -\alpha + \beta = 0, \qquad 4\alpha - \beta + \delta = -1, \qquad -3\alpha + 2\beta + \gamma + \delta = 0.$$

By solving this system we get:

$$\alpha = \beta = -1, \qquad \gamma = -3, \qquad \delta = 2,$$

so that the rate of spontaneous emission is

$$K = \frac{16\pi^3}{3} \frac{d^2}{\varepsilon_0 h \lambda^3}. \tag{3}$$

B.3 (0.7 pt)

$K = \dfrac{16\pi^3}{3} \dfrac{d^2}{\varepsilon_0 h \lambda^3}.$

B.4 (0.2 points)

By using the result of question B.2 and expressing transition dipole moment as $d = 2.4\,el$, we obtain from Eq. 3:

$$\tau_{\mathrm{Cy5}} = \frac{3}{16\pi^3} \frac{\varepsilon_0 h}{2.4^2 l^2 e^2} \lambda^3 \approx 3.3\,\mathrm{ns}.$$

B.4 (0.2 pt)

Numerical value: $\tau_{\mathrm{Cy5}} \approx 3.3\,\mathrm{ns}$.

Part C. Bose-Einstein condensation (1.5 points)

C.1 (0.4 points)

At temperature T, the average kinetic energy of translational motion is $\frac{3}{2}k_\mathrm{B}T$. Equating this result to $p^2/(2m)$, we obtain typical momentum $p = \sqrt{3mk_\mathrm{B}T}$ and the de Broglie wavelength

$$\lambda_\mathrm{dB} = \frac{h}{p} = \frac{h}{\sqrt{3mk_\mathrm{B}T}}.$$

C.1 (0.4 pt)

$p = \sqrt{3mk_\mathrm{B}T}$. $\lambda_\mathrm{dB} = \dfrac{h}{\sqrt{3mk_\mathrm{B}T}}$.

C.2 (0.5 points)

The volume per particle V/N is a good estimate for ℓ^3. We obtain $\ell = n^{-1/3}$, with $n = N/V$ and equate $\ell = \lambda_\mathrm{dB}$ to express $T_c = h^2 n^{2/3}/(3mk_\mathrm{B})$.

C.2 (0.5 pt)

$\ell = n^{-1/3}$. $T_c = \dfrac{h^2 n^{2/3}}{3mk_\mathrm{B}}$.

C.3 (0.6 points)

Using the answer to the previous question, we express $n_c = (3mk_\mathrm{B}T_c)^{3/2}/h^3$. Equation of state for the ideal gas gives $n_0 = p/(k_\mathrm{B}T)$. Numerical estimations yield $n_c \approx 1.59 \cdot 10^{18}\,\mathrm{m}^{-3}$ and $n_0/n_c \approx 1.5 \cdot 10^7$.

C.3 (0.6 pt)

Expression: $n_c = \dfrac{(3 \cdot 87\, m_\mathrm{amu} k_\mathrm{B} T_c)^{3/2}}{h^3}$. Numerical value: $n_c \approx 1.59 \cdot 10^{18}\,\mathrm{m}^{-3}$.

Expression: $n_0 = p/(k_\mathrm{B}T)$. Numerical value: $n_0/n_c \approx 1.5 \cdot 10^7$.

Part D. Three-beam optical lattices (5.0 points)

D.1 (1.4 points)

We sum the three electric fields (z components)

$$E(\vec{r},t) = E_0 \sum_{i=1}^{3} \cos\left(\vec{k}_i \cdot \vec{r} - \omega t\right), \tag{4}$$

and square the result

$$\begin{aligned} E^2(\vec{r},t) &= E_0^2 \sum_{i=1}^{3}\sum_{j=1}^{3} \cos\left(\vec{k}_i \cdot \vec{r} - \omega t\right)\cos\left(\vec{k}_j \cdot \vec{r} - \omega t\right) \\ &= \frac{E_0^2}{2} \sum_{i=1}^{3}\sum_{j=1}^{3} \left\{\cos\left[\left(\vec{k}_i - \vec{k}_j\right)\cdot\vec{r}\right] + \cos\left[\left(\vec{k}_i + \vec{k}_j\right)\cdot\vec{r} - 2\omega t\right]\right\}. \end{aligned} \tag{5}$$

Time averaging gives

$$\langle E^2(\vec{r},t)\rangle = \frac{E_0^2}{2} \sum_{i=1}^{3}\sum_{j=1}^{3} \cos\left[\left(\vec{k}_i - \vec{k}_j\right)\cdot\vec{r}\right], \tag{6}$$

we analyse the 9 terms and simplify to

$$\langle E^2(\vec{r},t)\rangle = E_0^2\left(\frac{3}{2} + \sum_{j=1}^{3} \cos\vec{b}_j \cdot \vec{r}\right). \tag{7}$$

Here $\vec{b}_{1,2,3} = (\vec{k}_2 - \vec{k}_3), (\vec{k}_3 - \vec{k}_1), (\vec{k}_1 - \vec{k}_2)$ or in terms of the Levi-Civita symbol, $\vec{b}_k = \varepsilon_{ijk}(\vec{k}_i - \vec{k}_j)$. Incidentally, they are known as the reciprocal lattice vectors.

D.1 (1.4 pt)

$V(\vec{r}) = -\alpha E_0^2\left(\frac{3}{2} + \sum_{j=1}^{3} \cos\vec{b}_j \cdot \vec{r}\right)$.

$\vec{b}_1 = \vec{k}_2 - \vec{k}_3, \qquad \vec{b}_2 = \vec{k}_3 - \vec{k}_1, \qquad \vec{b}_3 = \vec{k}_1 - \vec{k}_2.$

D.2 (0.5 points)

D.2 (0.5 pt)

Argument: Observe that rotation by 60° maps the three vectors $\vec{b}_{1,2,3}$ into the relabelled triplet of $-\vec{b}$'s.

D.3 (1.2 points)

We find

$$V(x,y) = -\alpha E_0^2 \left\{ \frac{3}{2} + \cos\left(ky\sqrt{3}\right) + \cos\left(\frac{3kx}{2} + \frac{ky\sqrt{3}}{2}\right) + \cos\left(\frac{3kx}{2} - \frac{ky\sqrt{3}}{2}\right) \right\}, \tag{8}$$

and deduce

$$V_X(x) = -\alpha E_0^2 \left\{ \frac{5}{2} + 2\cos\frac{3kx}{2} \right\}. \tag{9}$$

The potential has a simple cosine form, and the origin in an obvious minimum. Its replica appear at multiples of $\Delta x = 4\pi/(3k)$. In the midpoint between any two minima, e.g. at $x = \Delta x/2 = 2\pi/(3k)$, the function $V_X(x)$ has its maxima.

Concerning the behaviour along the y axis, we have

$$V_Y(y) = -\alpha E_0^2 \left\{ \frac{3}{2} + \cos 2\varphi + 2\cos\varphi \right\}, \qquad \varphi = \sqrt{3}ky/2. \tag{10}$$

Looking for the extrema, we find the equation

$$\sin 2\varphi + \sin\varphi = 0. \tag{11}$$

- $\varphi = 0$ is the 'deep' minimum – the lattice site;
- $\varphi = \pi$ is the 'shallow' minimum (later shown to be a saddle point of $V(x,y)$);
- $\varphi = 2\pi/3$ and $\varphi = 4\pi/3$ are maxima.

D.3 (1.2 pt)

$V_X(x) = -\alpha E_0^2 \left\{ \frac{3}{2} + 2\cos\frac{3kx}{2} \right\}$.

$V_Y(y) = -\alpha E_0^2 \left\{ \frac{3}{2} + \cos 2\varphi + 2\cos\varphi \right\}$, here $\varphi = \sqrt{3}ky/2$.

Minimum (-a) of $V_X(x)$: $x = 0$.
Maximum (-a) of $V_X(x)$: $x = \frac{2\pi}{3k}$.

Minimum (-a) of $V_Y(y)$: $y = 0$ ('deep') and $y = \frac{2\pi}{\sqrt{3}k}$ ('shallow').
Maximum (-a) of $V_Y(y)$: $y = \frac{4\pi}{3\sqrt{3}k}$ and $y = \frac{8\pi}{3\sqrt{3}k}$.

D.4 (0.8 points)

We review the minima found in the previous question and eliminate the saddle point at $(0, 2\pi/3\sqrt{3}k)$. The actual minima of the 2D potential landscape $V(x, y)$ are:

- $(0, 0)$ – at the origin;
- $(4\pi/(3k), 0)$ – nearest to the origin in the positive direction along the x axis. On the grounds of symmetry we argue that there are six equivalent nearest minima in the directions $0°$, $\pm 60°$, $\pm 120°$, and $180°$ with respect to the x axis.

Distance between nearest minima (the lattice constant) $a = 4\pi/(3k)$. Given that the laser wavelength is $\lambda_{\text{las}} = 2\pi/k$, we have $a = \Delta x = 2\lambda_{\text{las}}/3$.

D.4 (0.8 pt)

Ratio of the lattice constant to the laser wavelength: $a/\lambda_{\text{las}} = \dfrac{2}{3}$

Positions of all equivalent minima nearest to the origin: **in the directions 0°, ±60°, ±120°, and 180° with respect to the x axis**.

D.5 (1.1 points)

The atom's core electrons (all but the one promoted to to a state with a high principal quantum number n) shield the electric field of the nucleus so that the effective potential for the outer electron resembles that of a hydrogen atom. The attractive force acting on that electron, $F = e^2/(4\pi\varepsilon_0 r^2)$, gives rise to its centripetal acceleration $a = v^2/r$. Equating $F = m_e a$ and using the expression for the angular momentum $m_e v r = n\hbar$ to eliminate the velocity, we find the quantum number n corresponding to the orbit with the radius $r = \lambda_{\text{las}}$:

$$n = \frac{e}{\hbar}\sqrt{\frac{m_e \lambda}{4\pi\varepsilon_0}} \approx 85. \tag{12}$$

D.5 (1.1 pt)

Expression: $n = \dfrac{e}{\hbar}\sqrt{\dfrac{m_e\lambda}{4\pi\varepsilon_0}}$. Numerical value: $n \approx 85$.

Non-ideal capacitors (10 points)

Capacitance measurement method:

First, measure the highest voltage the capacitor can reach by connecting it to the voltage source via jumper wire W2. Before each measurement, connect capacitor to starting voltage source with jumper wire W2 and to a final voltage source (U_f) with jumper wire W1 via the resistor R1. Capacitor C2 should be prepared that way for at least 10 s, while C1 measurement can be started immediately by disconnecting jumper wire W2 from the starting voltage source. To determine a precise value of the final voltage U_f, it should be measured after capacitor has been connected to final source via R1 for a long time (at least 3 minutes). Then, the capacitance can be calculated from:

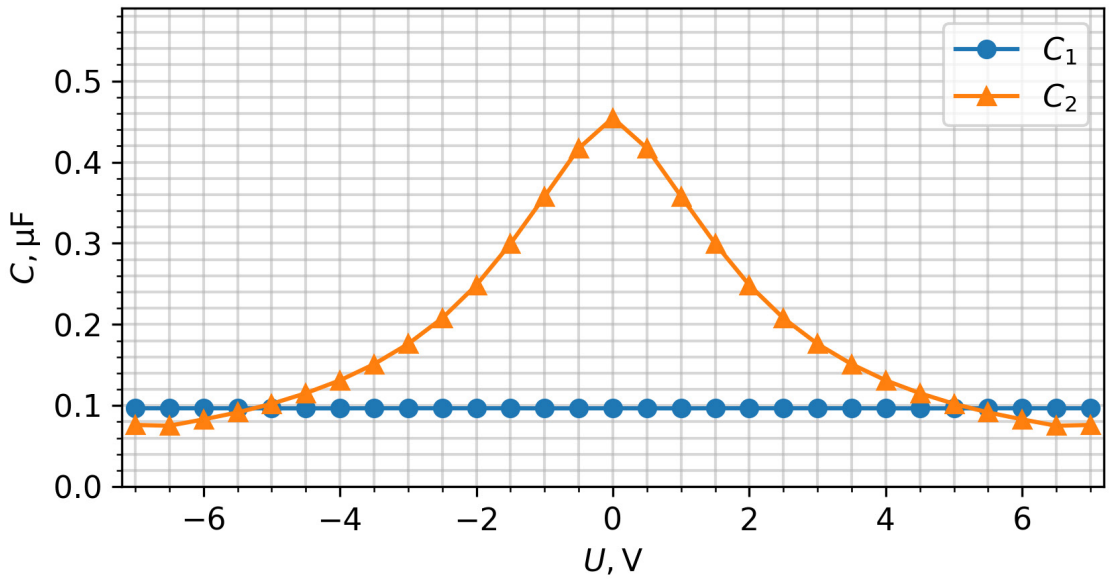
$$C(U) = \frac{U_f - U(t)}{R1} \Big/ \frac{\mathrm{d}U}{\mathrm{d}t}$$

When measuring C2, to ensure minimal change in charging current, capacitance should only be calculated in conditions where U_f and $U(t)$ have different polarities. This way, capacitance dependence on voltage should be symmetrical around 0 V.

Part A: Capacitors at room temperature (4 points)

A.1 (2.3 pt)

Graph $C_1(U)$ should be constant, $C_2(U)$ must be highest at 0 V.
Example results measured at room temperature of 29 °C.



	C_1	C_2
0 V	0.100 µF	0.473 µF
3 V	0.100 µF	0.183 µF
6 V	0.100 µF	0.086 µF

$$C(U) = \frac{U_f - U(t)}{R1} \Big/ \frac{\mathrm{d}U}{\mathrm{d}t}.$$

A.2 (0.5 pt)

$U_{\text{max change}} = 4.5$ V at capacitor C2

A.3 (1.2 pt)

It's important to calculate $\int_{0V}^{6V} C(U)dU$, not just attempt to multiply $C(6\text{ V}) \cdot 6\text{ V}$

$$q_1 = 0.60\ \mu\text{C}; \quad q_2 = 1.3\ \mu\text{C}$$

Part B: Calibrating NCT thermistor (1 point)

B.1 (1.0 pt)

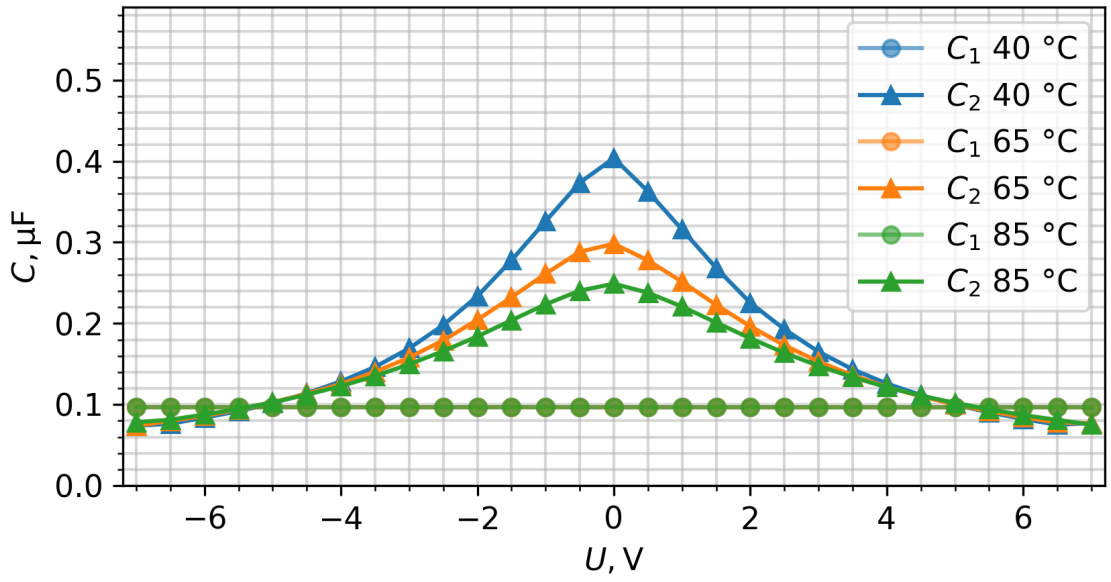
$R_0 = \frac{U_{T0}R_3}{U-U_{T0}} \mathrm{e}^{-B/T}$,
where $U = 3.3$ V, $U_{T0} - uT$ at room temperature, T – room temperature in kelvins

$R_0 = 0.0341\ \Omega$.

Part C: Capacitors at different temperatures (3 points)

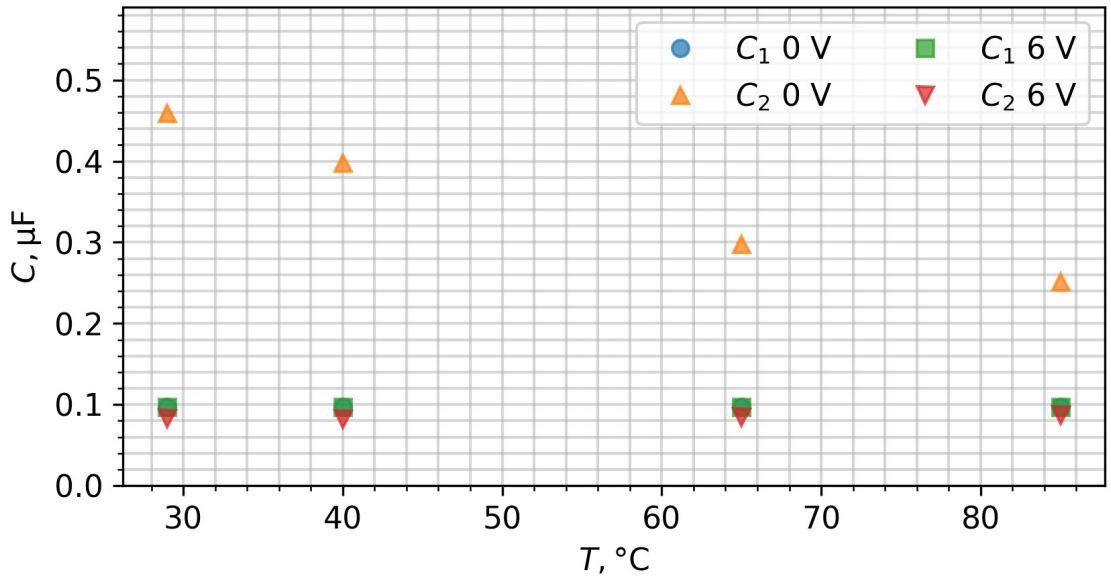
C.1 (1.3 pt)

Graphs $C_1(U,T)$ should always stay constant, $C_2(U)$ must be highest at 0 V



C.2 (0.5 pt)

Graph $C_1(T)$ should always stay constant



C.3 (1.2 pt)

$C_1(85\ °C)/C_1(40\ °C)\ |_{0V} = 1.00$

$C_1(85\ °C)/C_1(40\ °C)\ |_{6V} = 1.00$

$C_2(85\ °C)/C_2(40\ °C)\ |_{0V} = 0.63$

$C_2(85\ °C)/C_2(40\ °C)\ |_{6V} = 1.06$

Part D: Sources of measurement errors (2 points)

D.1 (1.0 pt)

Initial settings:

S1 position	IN connection
C1	-9V or GND

Process:

Step number	S1 position	IN connection	Duration, s	Measured variable
1	C1	+9V	0.2 s (any short time is good)	
2	C1	Free		$\lvert duC(t)\rvert/dt$
3	C1	+9V	5 s (has to be much longer than first)	
4	C1	Free		$\lvert duC(t)\rvert/dt$

Verification: $\lvert duC(t)\rvert/dt|_2 = \lvert duC(t)\rvert/dt|_4$

Main source of error: 1 (Leakage current.)

D.2 (1.0 pt)

Initial settings:

S1 position	IN connection
C2	-9V or GND

Process:

Step number	S1 position	IN connection	Duration, s	Measured variable
1	C2	+9V	0.2 s (any short time is good)	
2	C2	Free		$\lvert \mathrm{d}uC(t)\rvert/\mathrm{d}t$
3	C2	+9V	5 s (has to be much longer than first)	
4	C2	Free		$\lvert \mathrm{d}uC(t)\rvert/\mathrm{d}t$

Verification: $|\mathrm{d}uC(t)|/\mathrm{d}t|_2 \gg |\mathrm{d}uC(t)|/\mathrm{d}t|_4$

Alternatively,

$$\frac{|\mathrm{d}uC(t)|/\mathrm{d}t|_2}{|\mathrm{d}uC(t)|/\mathrm{d}t|_4} > 2.$$

Main source of error: 2 (Polarization properties of the capacitor's dielectric media)

Light Emitting Diodes (LEDs)

Volt-Ampere characteristics of the LED have to be measured in two modes: pulsed (part A) and continuous (part B). Running LED continuously produces a noticeable amount of heat, while running it in the pulsed mode allows minimizing and neglecting self-heating effect.

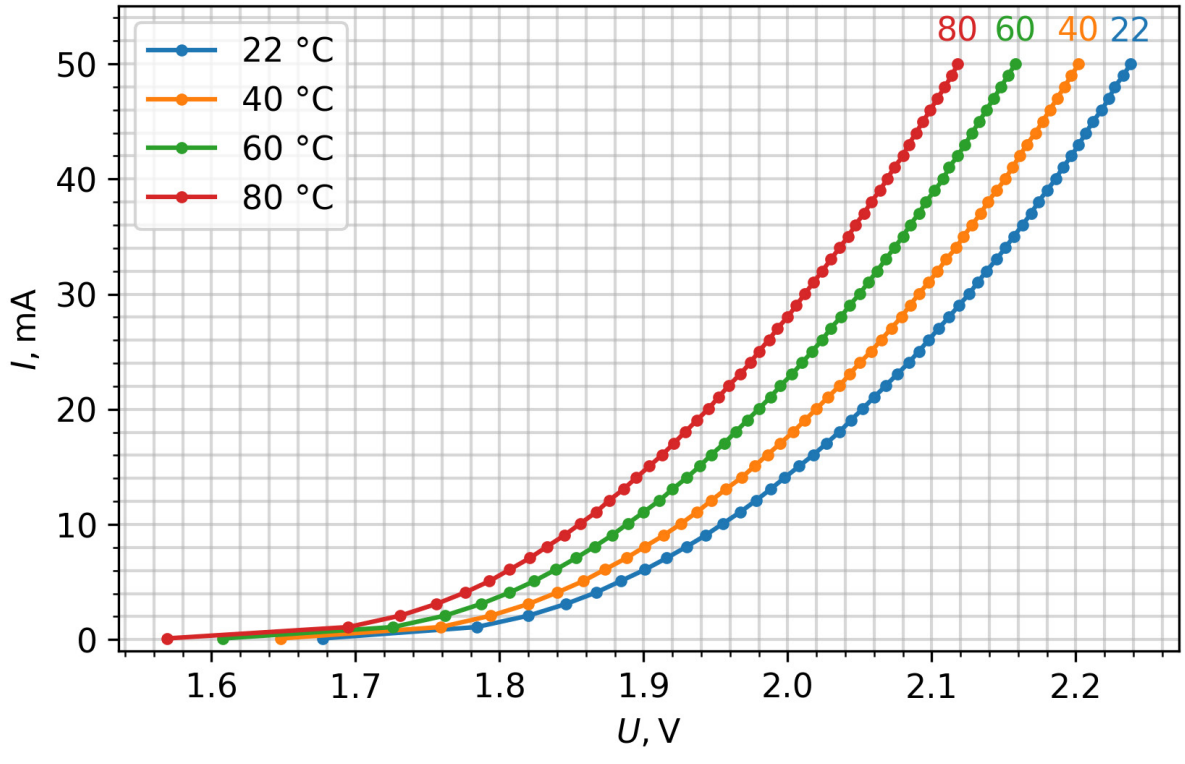
Students have to be able to run the automated $I_{\mathrm{LED}}(U_{\mathrm{LED}})$ measurement procedure and extract the required point by visually interpolating data for required values of I_{LED}.

The temperature of PCB is controlled by changing the current of the heating circuit. The heating and temperature measurement parts of this Experiment are identical to the Experiment 1.

Part A: Volt-ampere characteristics at different temperatures (5.0 points)

A.1 (2.5 pt.)

Graph $I_{\mathrm{LED}}(U_{\mathrm{LED}})$ has to be accurate (in right range) and smooth.



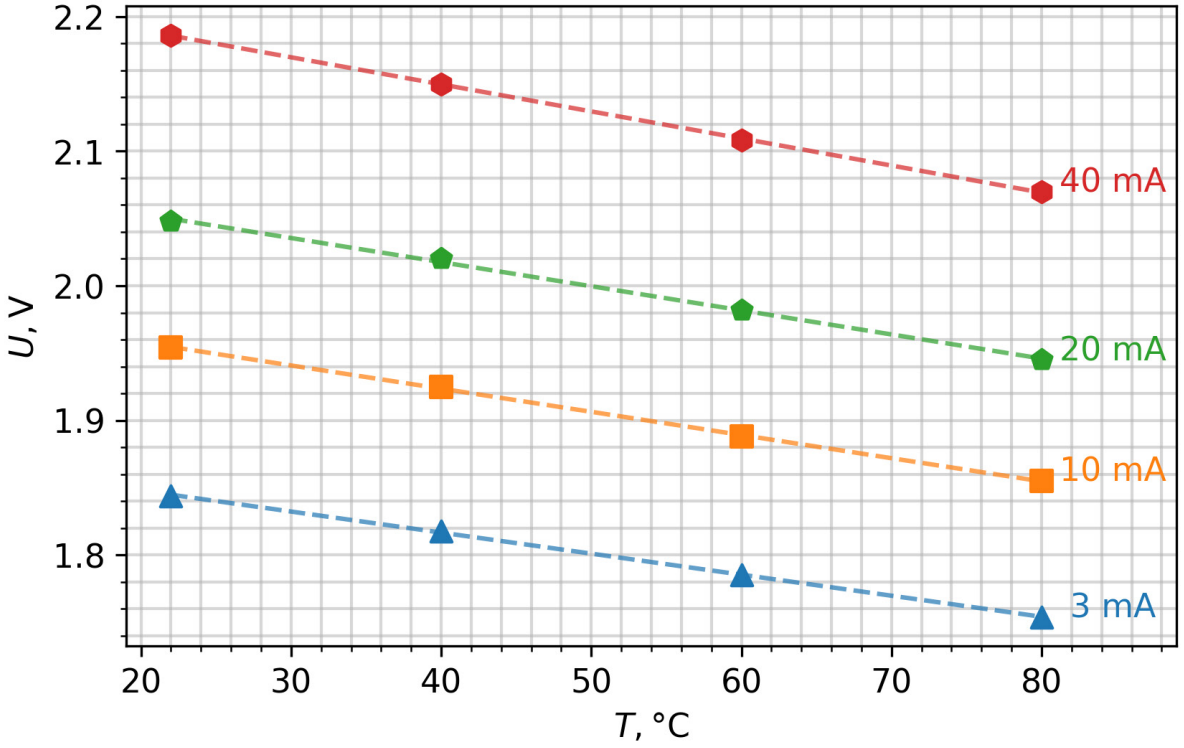
A.2 (1.0 pt.)

$U_{\mathrm{LED}}(I_{\mathrm{LED}}, T)$:

I_{LED} \ T	"Room" 22 °C	40 °C	60°C	80°C
3 mA	1.844 V	1.818	1.785	1.754
10 mA	1.954	1.925	1.888	1.855
20 mA	2.048	2.02	1.982	1.945
40 mA	2.186	2.15	2.108	2.07

A.3 (1.5 pt.)

Graphed $U_{\mathrm{LED}}(I_{\mathrm{LED}}, T)$ from A.2 data. $U_{\mathrm{LED}}(T)$ should show clear linear trend and be approximated graphically. The slope $\left(\frac{\Delta U(I,T)}{\Delta T}\right)$ should also be calculated.

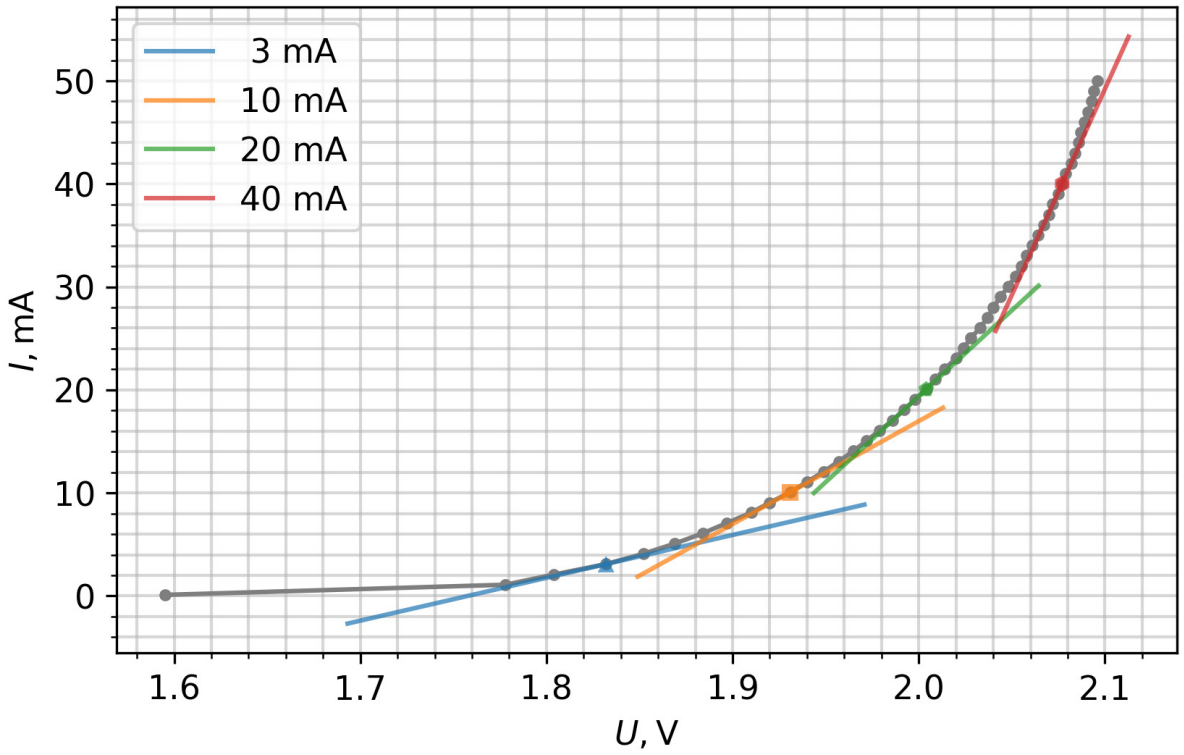


I_{LED}	3 mA	10 mA	20 mA	40 mA
$\left(\frac{\Delta U(I,T)}{\Delta T}\right)$	-1.55 mV/K	-1.7 mV/K	-1.8 mV/K	-2.0 mV/K

Part B: Measurement of the LED Volt-Ampere characteristics at continuous driving current (3.5 points)

B.1 (1.5 pt.)

Graph $I_{\mathrm{LED}}(U_{\mathrm{LED}})$ has to be accurate (in right range) and smooth.



I_{LED}	3 mA	10 mA	20 mA	40 mA
U_{LED}	1.83 V	1.93 V	2.00 V	2.08 V
ΔU	0.014 V	0.024 V	0.048 V	0.106 V
T_{J}	~ 32.3 °C	~43 °C	~49 °C	~76.5 °C
T_{PCB}	~ 25–30 °C	~ 30–35 °C	~ 33–37 °C	~ 35–40 °C

B.2 (0.5 pt.)

The dynamic resistance of the LED has to be calculated as derivative at the asked values of I_{LED}.

I_{LED}	3 mA	10 mA	20 mA	40 mA
$\frac{\mathrm{d}I}{\mathrm{d}U}$	41.6 mA/V	100 mA/V	166.7 mA/V	400 mA/V

B.3 (1.5 pt.)

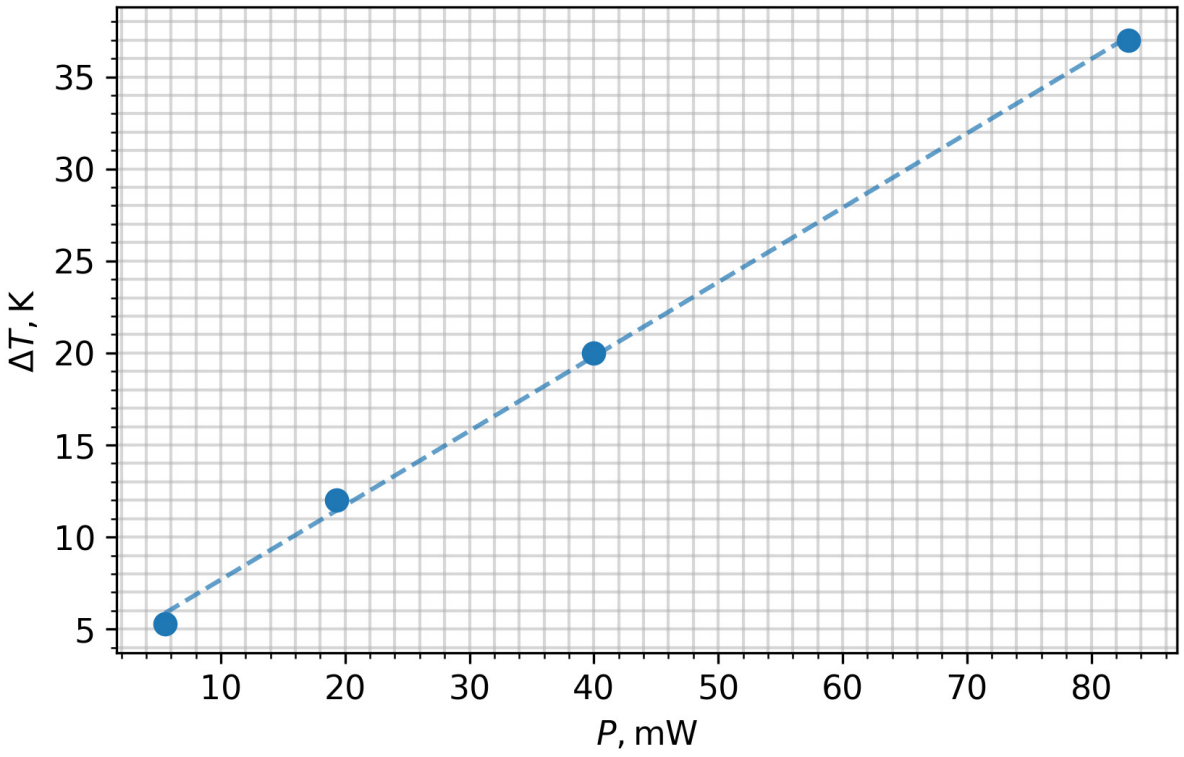
Graphed $\Delta T(P)$.

ΔT for each I_{LED} has to be calculated as $\Delta T = \frac{U(\mathrm{pulsed}) - U(\mathrm{CW})}{\left(\frac{\Delta U(I,T)}{\Delta T}\right)} - \left(T_{\mathrm{PCB}}(\mathrm{CW}) - T_{\mathrm{PCB}}(\mathrm{pulsed})\right)$ from the data of A.2, B.1 and A.3.

Caution: during the measurement of B.1, the temperature of the PCB is not constant and rises up to 7 °C above the "room' temperature at higher currents. This has to be taken into account when calculating ΔT.

The generated heat is taken as electrical power: $P = I_{\mathrm{LED}} \times U_{\mathrm{LED}}$. The energy emitted by the escaping light is neglected.

The graph should have a clear linear trend and approximated graphically. Thermal resistance is calculated as linear slope $\frac{\mathrm{d}}{\mathrm{d}P}\left(\Delta T(P)\right) = 400\ K/\mathrm{W}$.



I_{LED}	3 mA	10 mA	20 mA	40 mA
ΔT	5.0 K	12 K	20 K	37 K

Part C: Calculation of the LED current drift due to the temperature (1.5 points).

C.1 (1.5 pt)

The I_{LED} under constant $U_{\mathrm{LED}} = U_{20\mathrm{mA}}$ is calculated:

$$I_{LED}(U_{20\mathrm{mA}}, \mathrm{T}) = 20\ \mathrm{mA} - (T - T_{\mathrm{room}}) \times \left(\frac{\Delta U(20\ \mathrm{mA}, T)}{\Delta T}\right) \times \frac{\mathrm{d}I(20\ \mathrm{mA}, U)}{\mathrm{d}U}.$$

$I_{LED}(U_{20\mathrm{mA}}, 0°\mathrm{C}) = 13.3\ \mathrm{mA}$, $I_{LED}(U_{20\mathrm{mA}}, 40°\mathrm{C}) = 25.7\ \mathrm{mA}$.