CHN-S-1 T-1 C
Yongyan Han

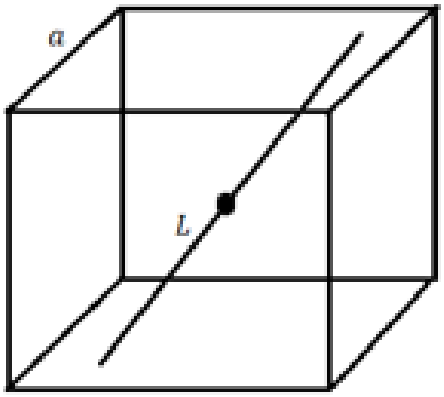
Theory
Q1
Cover sheet

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Problem mix (10 points)

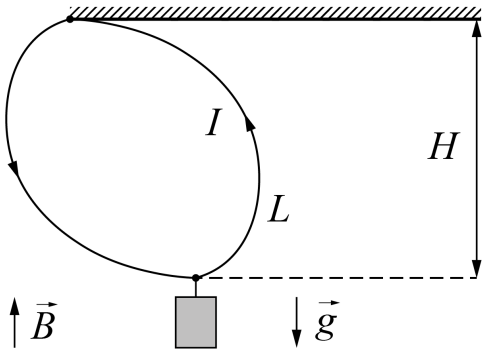
Part A. Cubic oscillations (2.5 points)

A.1 A narrow straight channel passes through the center of a fixed cube with a side a. The cube is uniformly charged, the charge density is ρ. The distance from the cube center to the point of intersection of the channel and a face is L. In the channel there is a particle of a mass m and a charge q. Find the period of small oscillations of the particle near the center. The gravitational interaction of the particle and the cube can be neglected. The cube and the particle are oppositely charged. 2.5pt



Part B. Suspension in magnetic field (3 points)

B.1 A current I flows through a loop made of a weightless flexible wire. The loop upper point is attached to the ceiling and a weight is suspended to its lowest point. The half length of the loop is L. The loop is placed in a vertical magnetic field B. The system has reached a stable equilibrium in which the point of suspension at the ceiling and the point of weight suspension are not on the same vertical. Find the wire tension T and the weight P if the distance from the ceiling to the lowest point of the loop is H. 3.0pt



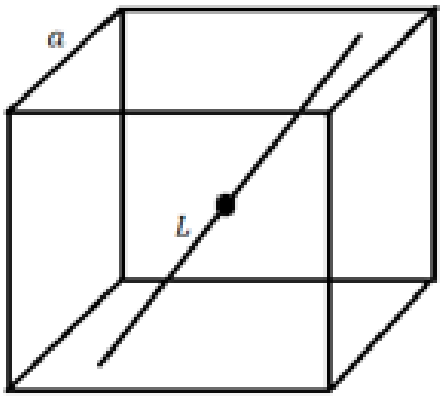
Part C. Rod in magnetic field (4.5 points)

C.1 A weightless rod of a length $2R$ is placed perpendicular to a uniform magnetic field $\vec{B}$. Two identical small balls of mass m and charge q each are attached at the rod ends. Let us direct z-axis along the magnetic field and place the origin at the rod center. The balls are given the same initial velocity v but in opposite directions so that one of the velocities is precisely in the z-direction. What are the maximum coordinates z_{max} of the balls? Express your answer in terms of q, B, m, v, and R. Find the magnitude of the ball accelerations at this moment and express your answer in terms of q, B, m, v, R, and z_{max}. 4.5pt

杂题（10 分）

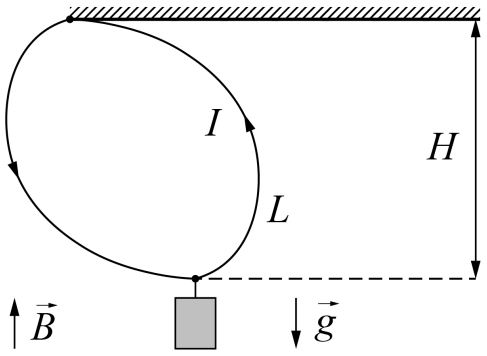
A 部分：立方体振荡（2.5 分）

A.1 一条狭窄的直线通道穿过一个边长为 a 的固定立方体的中心，立方体均匀带电，体电荷密度为 ρ，从立方体中心到通道与面的交点的距离为 L，通道中含有一个质量为 m、电荷量为 q 的粒子，求该粒子在中心附近的小振动周期。粒子与立方体的引力作用可以忽略。立方体和粒子带电的电荷种类相反。 2.5pt



B 部分：磁场悬浮 (3 分)

B.1 一电流 I 流过一个没有重量的软导线形成一个回路。其上端连接在天花板上，下端悬挂一个重物。回路一半的长度为 L 。上述体系在竖直磁场 B 的影响下，建立了一个稳定的平衡，其中天花板的悬挂点和重物的悬挂点不在同一竖直方向上。若天花板到回路底部的距离为 H，求软导线的张力 T 和重物的重量 P。 3.0pt



C 部分：磁场中的杆 (4.5 分)

CHN-S-1 T-1 Q-2

Q1-2

Chinese (People's Republic of China)

C.1	一根长度为 2R 的无重量杆垂直于匀强磁场 $\vec{B}$ 的方向．在杆的两端固定有 2 个质量均为 m，电荷量均为 q 的相同小球。现取 z 轴与磁场方向一致，其原点在棒的中心。现分别给予 2 小球大小相同、方向相反的初速度 v，其中一个小球的初速度方向与磁场方向相同。求小球的最大坐标 zmax，并用 q,B,m,v,R 表示。求出此刻球的加速度大小，并用 q,B,m,v,R 和 zmax 表示。	4.5pt

杂题 (10 points)

A.1 (2.5 pt)

$T =$

B.1 (3.0 pt)

$T =$

$P =$

C.1 (4.5 pt)

$z_{\text{max}} =$

$a =$

W1-1

W1-2

W1-3

W1-4

W1-5

W1-6

W1-7

W1-8

W1-9

W1-10

W1-11

W1-12

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W1-13

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W1-14

W1-15

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Theory
Q2
Cover sheet

Please return this cover sheet together with all the related question sheets.

Anisotropic friction (10 points)

Force of friction is not always isotropic. Often the magnitude and direction of a friction force depend on the direction of body motion. For example, friction anisotropy can arise in the presence of "grooves" of a certain orientation on the contact surface of bodies (it is known that the coefficient of friction of oak against oak along and across the grain is equal to 0.48 and 0.34, respectively). Friction anisotropy can lead to unusual properties of motion which are studied in this problem.

What you need to know about anisotropic friction

Suppose that a surface is made of an anisotropic material. One of the most popular models of anisotropic friction suggests that there are perpendicular axes X and Y (they are called *primary*) so that the friction force $\vec{F}$ acting on a body will depend on the direction of the body motion as

$$\begin{aligned} F_x &= -\frac{|N|}{|v|}\mu_x v_x \\ F_y &= -\frac{|N|}{|v|}\mu_y v_y \end{aligned} \tag{1}$$

where F_x and F_y are the friction force components, N is a normal reaction force acting on the body, v_x and v_y are components of the velocity vector $\vec{v}$, and μ_x and μ_y are the friction coefficients along the primary axes.

Hereinafter, it is understood that the coordinate axes on the plane coincide with the primary axes. The friction coefficients are $\mu_x = 0{,}75$ and $\mu_y = 0{,}5$ unless otherwise stated.

In parts A and B a body can be considered as point-like. The plane, on which the bodies move, is horizontal in all parts of the problem.

Provide a numerical answer wherever possible.

Part A. Motion on a horizontal surface (4.0 points)

A.1	At what angle α_1 to the X axis should the body velocity vector be for the absolute value of the power of the friction force be at maximum?	0.5pt

A.2	At what angle α_2 to the X axis should the body velocity vector be for the absolute value of the power of the friction force be 1.2 times less than maximum?	0.5pt

A.3	Let the initial velocity have components $v_{0x} = 1\ \mathrm{m/s}$ and $v_{0y} = 1\ \mathrm{m/s}$. After some time the velocity component along the Y axis equals $v_{1y} = 0{,}25\ \mathrm{m/s}$. What is the velocity magnitude at this moment?	1.0pt

A.4	Let the velocity be $v_2 = 1.0\ \mathrm{m/s}$. At what angle α_3 to the X axis should the velocity vector be for the radius of curvature of the trajectory be minimum? What is this radius equal to? The free fall acceleration is $g = 9{,}8\ \mathrm{m/s^2}$.	1.0pt

A.5	In a single diagram on the XY plane, sketch the trajectories of the body launched at the angles $\alpha_4 = \pi/6$ and $\alpha_5 = \pi/3$ for the friction coefficients specified above. The magnitudes of initial velocities are the same. Solve the same problem for the friction coefficients $\mu_x = 0{,}4$ and $\mu_y = 0{,}7$.	1.0pt

Part B. Conditions for the beginning of body movement (2.0 points)

B.1	A body of mass m is at rest at the origin. A force has been applied to it at an angle α to the X axis. The force magnitude $F(t) = \gamma t$ linearly grows with time. Find the dependence of the moment the body starts moving on α. Ignore the stagnation phenomenon.	2.0pt

Part C. Circular motion (4.0 points)

Two identical point-like masses m are connected with a weightless inextensible rod of length $L = 1\ \mathrm{m}$, the system lies on a surface with anisotropic friction. The rod is aligned with the Y axis and does not touch the surface. One of the masses is given an initial velocity perpendicular to the rod.

C.1	For a given initial velocity v_0 find the dependence of its velocity v on the angle of rotation of the rod φ assuming that the other body remains at rest.	1.5pt

C.2	Find the maximum value of the initial velocity $v_{0\mathrm{max}}$ at which the other body will remain at rest.	1.5pt

C.3	What distance will the body travel until it stops completely if the initial velocity is $v_{0\mathrm{max}}$?	1.0pt

各向异性摩擦 (10 分)

摩擦力并不总是各向同性的。通常，摩擦力的大小和方向可能取决于物体的运动方向。例如，在物体的接触面上存在固定方向的” 沟槽” 时，会产生摩擦各向异性（已知橡木对橡木沿木纤维纹理方向的摩擦系数等于 0.48，而橡木对橡木沿着纤维交叉方向的摩擦系数为 0.34）。摩擦力中各向异性的存在会导致运动的异常特性，本问题研究的就是这个特性。

关于各向异性摩擦力，你需要知道如下内容

假设一个表面是由一种各向异性材料制成的。其中一个最流行的各向异性摩擦力模型表明，存在相互垂直的轴 X 和 Y（称为 主轴），使得作用在物体上的摩擦力 $\vec{F}$, 取决于物体的运动方向，具体如下：

$$\begin{aligned} F_x &= -\frac{|N|}{|v|}\mu_x v_x \\ F_y &= -\frac{|N|}{|v|}\mu_y v_y \end{aligned} \tag{1}$$

其中 F_x，F_y 是摩擦力的分量，N 是作用在物体上的法向正压力，v_x 和 v_y 是速度矢量 $\vec{v}$ 的分量，μ_x 和 μ_y 是沿主轴方向的摩擦系数。

下面假设平面上的坐标轴与主轴重合。除非另有说明，摩擦系数为 $\mu_x = 0.75$，$\mu_y = 0.5$。

在 A、B 两部分中，可以认为物体是质点。在问题的所有部分中，物体运动的平面是水平的。

尽可能提供一个数字答案。

A 部分. 物体在水平面上的运动 (4.0 分)

A.1	物体的速度矢量应与 X 轴成多大角度 α_1，摩擦力功率的绝对值才会最大?	0.5pt
A.2	物体的速度矢量与 X 轴成多大角度 α_2，可以使摩擦力功率绝对值的 1.2 倍小于最大值?	0.5pt
A.3	设物体的初速度的投影为 $v_{0x} = 1$ m/s，$v_{0y} = 1$m/s。一段时间后，速度在 Y 轴上的投影等于 $v_{1y} = 0.25$m/s。此时物体速度的大小是多少?	1.0pt
A.4	设物体的速度大小等于 $v_2 = 1.0$ m/s。速度矢量与 X 轴成什么角度 α_3 时，物体运动轨迹的曲率半径最小? 这个半径等于多少? 自由落体加速度为 $g = 9.8$ m/s^2。	1.0pt
A.5	对于前面指出的摩擦系数，在一张图上定性地勾画出发射角分别为 $\alpha_4 = \pi/6$ 和 $\alpha_5 = \pi/3$ 时，物体在 XY 平面的轨迹。初速度的大小都相同。如果摩擦系数为 $\mu_x = 0.4$，$\mu_y = 0.7$，求同样的问题。	1.0pt

B 部分. 物体开始运动的条件 (2.0 分)

B.1	一个质量为 m 的物体静止在原点。对其施加一个力，并与 X 轴成角 α，力的大小 $F(t) = \gamma t$ 与时间成正比。忽略停滞效应，求出该物体从原点开始运动时的时间时刻 t 与角度 α 的关系。	2.0pt

C 部分 圆周运动（4.0 分)

两个质量为 m 的可近似为质点的物体，位于各向异性摩擦力的表面上，并由长度为 L=1m 的无质量不可伸缩杆连接。杆沿着 Y 轴方向放置，并且不与表面接触。给其中一个物体一个垂直于杆的初始速度。

C.1	设定物体的初始速度为 v_0，求其后续速度 v 与杆的旋转角度 φ 的关系，假设后续运动中另一物体始终处于静止状态。	1.5pt
C.2	求出初始速度的最大值 $v_{0\max}$，在这个值上，另一个物体仍将保持静止状态。	1.5pt
C.3	当物体以初始速度 $v_{0\max}$ 运动时，从开始到完全停止该物体移动的距离是多少？	1.0pt

各向异性的摩檫力 (10 points)

A.1 (0.5 pt)

$\alpha_1 =$

A.2 (0.5 pt)

$\alpha_2 =$

A.3 (1.0 pt)

$v =$

A.4 (1.0 pt)

$\alpha_3 =$

$R =$

A.5 (1.0 pt)

B.1 (2.0 pt)

$t(\alpha) =$

C.1 (1.5 pt)

$v(\varphi) =$

C.2 (1.5 pt)

$v_{0\mathrm{max}} =$

C.3 (1.0 pt)

$s =$

W2-2

W2-3

W2-4

W2-6

W2-7

W2-8

W2-9

W2-10

W2-11

W2-12

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W2-13

W2-14

CHN-S-1 T-2 W-15

W2-15

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CHN-S-1 T-3 C
Yongyan Han

Theory
Q3
Cover sheet

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Laser technologies (10 points)

When solving the problem, use the following values of the physical constants:

the speed of light in vacuum $c = 3.00 \cdot 10^8$ m/s;

Planck's constant $\hbar = 1.055 \cdot 10^{-34}$ J · s;

constant in Coulomb's law and electric constant $k = \frac{1}{4\pi\varepsilon_0} = 8.99 \cdot 10^9 \ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}$;

elementary charge $e = 1.60 \cdot 10^{-19}$ C.

Part A. Classical superradiance model

A laser is a source of coherent optical radiation. Laser radiation is generated when a large number of atoms transferred to an excited state by external action (pumping) emit photons with the same phase and polarization. A consistent theory of laser radiation is based on quantum mechanics but some aspects of this phenomenon can be understood by means of classical electrodynamics.

Let us first consider emission of a photon by a single atom. According to classical electrodynamics, an atom can be regarded as dipole emitter. In this model an electric dipole is associated with an atom comprised of an immobile atomic nucleus with a positive charge $+q$ and a negative charge $-q$ oscillating harmonically around it (the negative charge is located at the center of the charge distribution of electron cloud).

Here the atomic dipole moment oscillates according to the law $\vec{p}(t) = \vec{p}_m \cos(\omega t + \varphi)$. The cyclic oscillation frequency is related to the energy of emitted photons by the Planck relation $E_\gamma = \hbar\omega$. Hereinafter the frequency of photons means cyclic frequency. The radiation power of a classical system with a variable dipole moment $\vec{P}(t)$ is given by the formula

$$W = \frac{2k}{3c^3}\left\langle \left(\frac{d^2\vec{P}}{dt^2} \right)^2 \right\rangle, \tag{1}$$

where the angle brackets stand for the averaging over the oscillation period.

A.1 An atom emits light with a wavelength $\lambda_0 = 300$ nm. Using the classical model estimate an emission time τ (that is, the period of time it takes the atom to emit the energy equal to that of a single photon). This time coincides with the characteristic time, during which the atom emits a photon, by the order of magnitude. All radiation is due to a single electron located at a distance about $a_0 = 0.1$ nm from the nucleus. Express your answer in terms of the physical constants, λ_0, and a_0. 1.0pt

Suppose N atoms in a certain volume are transferred to an excited state by a short-term pumping action. It is known that one atom emits a photon with a frequency ω for a characteristic time τ.

A.2 Estimate the power W_s of electromagnetic radiation of all N atoms in the spontaneous emission mode, i.e. when the direction of atomic dipole and the phase of its oscillations randomly change from atom to atom. In your answer write down the formula for the power in terms of N, ω, and τ. 0.25pt

A.3 Estimate the duration of the spontaneous emission pulse of this system of atoms. Express your answer in terms of the same quantities. 0.25pt

A.4	Estimate the power W_i of electromagnetic radiation of all N atoms in the superradiance mode, i.e. when the direction of atomic dipoles and the phases of their oscillations are the same for all atoms in the excited state. Express your answer in terms of N, ω, and τ.	0.5pt

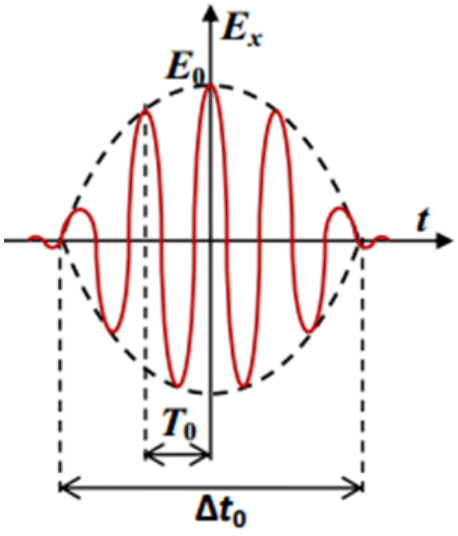
A.5	Estimate the duration of the radiation pulse of the system of atoms in the superradiance mode. Express your answer in terms of the same quantities.	0.25pt

Part B. Nonlinear optics and pulse compression

Pulses of even shorter duration can be obtained by reducing the duration of already generated laser pulses. A pulse duration Δt and a dispersion of frequency of pulse oscillations $\Delta\omega$ (spectral width) are related by the inequality $\Delta\omega\Delta t \geq 2\pi$. Laser pulses generated in the superradiance mode already have the shortest possible duration for the given dispersion of frequencies, $\Delta t_0 \approx \frac{2\pi}{\Delta\omega_0}$. Therefore, the pulse duration can be reduced in two steps: first, increase the spectral width of the pulse (without changing the duration) and second, compress the pulse in time.

One of the most common ways to solve the first problem is pulse chirping. This method is based on the use of nonlinearity of a medium, i.e. dependence of the refractive index of the medium n on the amplitude of oscillations of the electric field E_m of the wave. The dependence is of the form $n = n_0 + n_2 E_m^2$, where n_0 and n_2 are some constants specific for the medium. Nonlinear effects are small, e.g. in quartz at a light intensity $I_1 = 10^9$ W/cm^2 the refractive index increases only by $n_2 E_{m1}^2 \approx 3.2 \cdot 10^{-7}$. The intensity of an electromagnetic wave in a medium is determined by the formula $I = \frac{\varepsilon_0 n_0 c}{2} E_m^2$.

Consider a pulse of duration Δt_0 with a small dispersion of frequencies $\Delta\omega_0 \approx \frac{2\pi}{\Delta t_0}$, the average pulse frequency is ω_0. An approximate dependence of the electric field on time in such a pulse is shown in the Figure. The speed of the wave maxima is the same at the pulse edges and in the central part it decreases due to the nonlinearity effects. Because of that the total pulse duration does not change, the frequency increases at the rear part of the pulse and decreases at the front. Such a pulse is called "chirped".



B.1	Let the amplitudes of two wave maxima be E_{m1} and E_{m2}. Find the difference in their propagation speeds Δv. Express your answer in terms of n_0, n_2, c, E_{m1}, and E_{m2}.	0.5pt

B.2	A light pulse with a wavelength in vacuum of $\lambda_0 = 300$ nm and a maximum intensity of $I_0 = 3 \cdot 10^9$ W/cm^2 propagates along the axis of a quartz fiber. Assume the envelope of a time dependence of the electric field squared $E_m^2(t)$ of the wave to be a parabola. How far (find the distance s) does the pulse propagate along the fiber before its spectral width increases by the factor of $K = 200$? Express your answer in terms of K, λ_0, n_2, E_m and calculate the numerical value (in meters, rounded to an integer).	2.0pt

To compress a chirped pulse in time one can pass it through a medium in which the group velocity of the wave depends on its frequency. For the medium under consideration, the dependence of wavenumber on frequency in the vicinity of the mean frequency ω_0 can be represented as $k(\omega) = k_0 + \beta_1(\omega - \omega_0) + \frac{\beta_2}{2}(\omega - \omega_0)^2$, where $\beta_1 = 5$ ns/m and $|\beta_2| = 20$ fs^2/mm.

B.3	What sign should the constant β_2 have in order for the pulse chirped according to the scheme described above to be compressed in time in this medium? Please, indicate "+" or "-" in your answer. In what follows consider that β_2 has exactly this sign.	0.5pt

B.4	A pulse described in B2 has a duration $\Delta t_0 = 10$ ps and an initial spectral width $\Delta\omega_0 \approx 2\pi/\Delta t_0$ (before chirping) and propagates in the medium described above. Find the distance the pulse should travel in order to achieve the minimum possible duration after chirping with spectrum broadening by the factor of $K = 200$. Express your answer in terms of physical constants, $K, \Delta t_0, \beta_1$, and β_2 and calculate the numerical value in meters, rounded to an integer.	1.0pt

B.5	Nonlinearity of a medium leads to disappearance of diffraction of a light beam of sufficiently high intensity. Estimate the minimum power of a light pulse W_c at which it does not experience diffraction, i.e. propagates inside a narrow cylindrical channel of constant radius. Express your answer for W_c in terms of physical constants, frequency ω_0, n_0, and n_2. Assume the intensity distribution over the channel cross section to be approximately uniform. Find the numerical value of the power for a pulse with a wavelength in vacuum $\lambda_0 = 300$ nm propagating in quartz. Coefficient $n_0 = 1.47$.	1.5pt

Part C. Exoplanets

In astronomy observations of luminous objects are being made for long periods of time. This makes it possible to study variations of their emission spectra. Spectral measurements can detect planets orbiting distant stars -- "exoplanets". Exoplanets do not have their own radiation so one is bound to study radiation spectra of their stars. If the line of sight from Earth to an exoplanet lies almost in its orbital plane, such an exoplanet can be discovered by a decrease of the star brightness at a moment the exoplanet is crossing the star disk. However, if the orbital plane is tilted with respect to the direction to Earth this method does not work.

Q3-4

English (Official)

C.1	Propose a method that would allow one to detect an exoplanet with a noticeable inclination of its orbital plane with respect to the line of sight by means of studying the spectrum of its star in the optical range. As an answer name the physical phenomenon underlying your method.	1.0pt
C.2	Suppose an exoplanet of mass m orbits a star of mass M in a circular orbit of a radius R and the period of revolution is T. The orbital plane is at an angle θ to the direction to Earth. Estimate the accuracy of the relative frequency measurement, $\Delta\omega/\omega$, required to detect such an exoplanet by your method. In your answer express $\Delta\omega/\omega$ in terms of the fundamental constants, R, T, θ, m, and M.	1.0pt
C.3	Assume the mass of the exoplanet and its star to be equal to the mass of Earth and the Sun, respectively. Assume the radius of the circular orbit to be equal to the distance from Earth to the Sun ($R \approx 1.5 \cdot 10^{11}$ m), the angle $\theta = 60^\circ$. The Solar mass is 330,000 times of the Earth's mass, the period of the Earth's revolution around the Sun is 1 year. Find an integer n such that 10^{-n} is the accuracy of relative frequency measurement required by your method. Usage of ultrashort (femtosecond) laser pulses makes it possible to measure frequencies in the optical range (10^{15} Hz) with an accuracy of about 10 Hz. Is this accuracy enough to register the exoplanet?	0.25pt

激光技术

解题时，请用下列物理常数的值。

真空中的光速 $c = 3.00 \cdot 10^8$ m/s

约化普朗克常数 $\hbar = 1.055 \cdot 10^{-34}$ J · s。

库仑定律中的常数和电常数 $k = \frac{1}{4\pi\varepsilon_0} = 8.99 \cdot 10^9 \ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}$。

基本电荷 $e = 1.60 \cdot 10^{-19}$ C。

A 部分. 经典超辐射模型

激光器是相干光辐射的光源。激光辐射的形成是由于大量原子在外部作用（泵浦）下转入激发态，发射出具有相同相位和偏振的光子。要建立一个一致的激光辐射理论，需要使用量子力学，但其工作的某些方面可以借助于经典电动力学来理解。

让我们首先考虑单个原子发射光子的情况。从经典电动力学的角度来看，一个原子可以被看作是一个电偶极发射体。在这个模型中，一个电偶极子与原子相关联，由一个不动的带正电荷的原子核形成，它的电荷为 +q，负电荷为-q，在它周围简谐地振动（位于电子云的电荷分布中心）。在这种情况下，原子的偶极矩根据规律 $\vec{p}(t) = \vec{p}_m \cos(\omega t + \varphi)$ 变化。循环振荡频率与发射光子的能量由普朗克关系 $E_\gamma = \hbar\omega$ 相关。在此，光子的频率指角频率。一个具有可变偶极矩的经典系统的辐射功率 $\vec{P}(t)$ 由公式决定。

$$W = \frac{2k}{3c^3}\left\langle \left(\frac{d^2\vec{P}}{dt^2}\right)^2 \right\rangle, \tag{1}$$

其中，角括号内为振荡周期的平均值。

A.1	原子发出的光波长为 $\lambda_0 = 300$ nm。利用经典模型，估计原子的发射时间 τ（即原子发射能量等于一个光子能量所用的时间）。这个时间与原子发射一个光子的特征时间在数量级上相吻合。所有的辐射都是由距离原子核约 a_0=0.1 nm 的一个电子产生的。请用物理常数 λ_0 和 a_0 来表示你的答案。	1.0pt

假设在一定体积内，N 个原子通过短期泵浦作用转入激发态。已知一个原子在特征时间 τ 内发射一个角频率为 ω 的光子

A.2	请根据自发辐射模型，即不同原子的偶极子的方向及其振荡相位为随机变化，估计所有 N 个原子的自发电磁辐射功率 W_s。在你的答案中，以 N, ω 和 τ 为单位写出答案。	0.25pt
A.3	估算该原子系统自发发射脉冲的持续时间。请用相同的物理量来表示你的答案。	0.25pt
A.4	估算所有 N 个原子在超辐射模式下的电磁辐射功率 W_i，即所有原子在受激状态下的原子偶极子方向及其振荡相位相同时的辐射功率。以 N, ω 和 τ 为单位写出答案。	0.5pt
A.5	估计超辐射模式下原子系统辐射脉冲的持续时间。用相同的物理量表示你的答案。	0.25pt

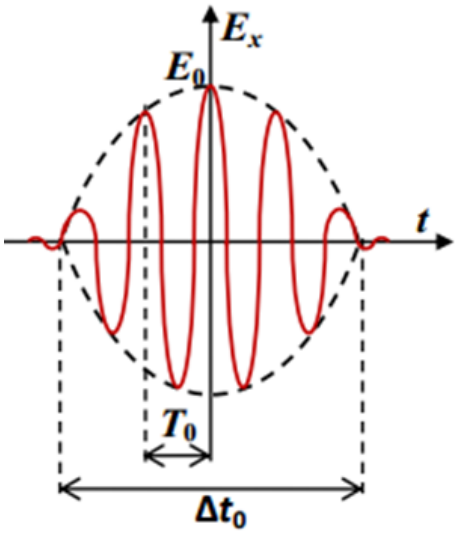
B 部分. 非线性光学和脉冲压缩

通过压缩已经产生的激光脉冲的持续时间，可以得到持续时间更短的脉冲。脉冲持续时间 Δt 和脉冲的光谱宽

度 $\Delta\omega$ 由不等式 $\Delta\omega\Delta t \geq 2\pi$ 表示。在超辐射模式下产生的激光脉冲，在给定的光谱宽度下，$\Delta t_0 \approx \frac{2\pi}{\Delta\omega_0}$ 为脉冲的最短的持续时间。因此，脉冲持续时间可以分两个阶段压缩：首先，增加脉冲的光谱宽度（不改变持续时间），然后在时间上对脉冲进行压缩。

解决第一个问题的最流行的方法之一是脉冲啁啾（来自英文”chirp”）。这种方法的原理是利用介质的非线性光学效应，即介质的折射率 n 对电场强度振幅 E_m 的依赖性。这种依赖关系的形式为 $n = n_0 + n_2 E_m^2$，其中 n_0、n_2 是取决于物质性质的常数。非线性效应很小，例如，在光照强度 $I_1 = 10^9$ W/cm^2 的石英中，折射率只增加 $n_2 E_{m1}^2 \approx 3.2 \cdot 10^{-7}$。介质中电磁波的强度由公式 $I = \frac{\varepsilon_0 n_0 c}{2} E_m^2$ 决定。

考虑一个持续时间为 Δt_0 的脉冲，有很小的光谱宽度 $\Delta\omega_0 \approx \frac{2\pi}{\Delta t_0}$，中心频率为 ω_0。图中显示了这种脉冲中电场强度对时间的近似依赖关系。脉冲边缘的波峰运动速度是相同的，在中央部分由于非线性的影响而减小。由于这个原因，脉冲的总持续时间没有变化，在脉冲的” 后部” 频率增加，而在” 前部” 频率减少。这种脉冲称为” 啁啾的”。



B.1	设两波最大值的振幅分别等于 E_{m1} 和 E_{m2}。求它们的传播速度之差 Δv。用 n_0, n_2, c, E_{m1} 和 E_{m2} 来表示你的答案。	0.5pt

B.2	在真空中波长为 $\lambda_0 = 300$nm，最大强度为 $I_0 = 3 \cdot 10^9$ W/cm^2 的光脉冲沿石英光纤的轴线传播。将波的电场强度的平方的时间依赖的包络线 $E_m^2(t)$ 看作是一条抛物线。请确定光纤的长度 s，在通过期间，使得脉冲的光谱宽度增加 K=200 倍。用 K, λ_0, n_2, E_m 表示你的答案，也可计算出数值（单位为米，四舍五入为整数值）。	2.0pt

为了在时间上压缩一个啁啾脉冲，可以让它通过一个介质，在这个介质中，波的群速取决于频率。在平均频率 ω_0 附近，对于所考虑的介质，波矢对频率的依赖可以表示为 $k(\omega) = k_0 + \beta_1(\omega - \omega_0) + \frac{\beta_2}{2}(\omega - \omega_0)^2$。对于所用介质取 $|\beta_2| = 20$ fs^2/mm。

B.3	按上述方案进行的脉冲压缩，在该介质中要想在时间上压缩该脉冲，请确定 β_2 的符号应该是什么？请在答案中用”+” 或”-” 表示。在下面的内容中，请使用所确定的 β_2。	0.5pt

B.4	从 B2 发出的脉冲，它的持续时间 $\Delta t_0 = 10$ ps，初始光谱宽度 $\Delta\omega_0 \approx 2\pi/\Delta t_0$(加啁啾前)，在上面的介质中需要传播多大距离以达到加啁啾后的最小持续时间，光谱展宽倍数为 $K = 200$。用物理常数 $K, \Delta t_0, \beta_1$ 和 β_2 来表示你的答案，并计算出以米为单位的数值，四舍五入为整数。	1.0pt

B.5	足够高强度光束下，介质的非线性效应将抵消衍射导致的光束发散。估算光脉冲的最小功率 W_c，在该功率下它不发生光束发散，即它在一个半径不变的窄圆柱形通道内传播。用频率 ω_0、n_0 和 n_2 来表达你对 W_c 的解答。认为通道截面上的强度分布近似均匀。求在真空中波长为 $\lambda_0 = 300$ nm 的脉冲，在石英中传播的该功率的数值。系数 $n_0 = 1.47$。	1.5pt

C 部分. 系外行星

在天文学中，对发光物体的观测是长时间进行的。这使得研究其发射光谱的变化成为可能。光谱测量可以探测围绕其他恒星运行的行星 – –” 系外行星”。外行星没有自己的辐射，所以你必须研究其恒星的辐射光谱。如果一颗系外行星对地球的方向几乎是在其轨道的平面上，那么根据系外行星通过恒星盘的瞬间恒星亮度的降低，就可以发现这样一颗系外行星。但是，如果轨道平面相对于对地球的方向是倾斜的，这种方法起不到作用。

C.1	提出一种方法，通过研究一颗恒星在光学范围内的光谱，可以探测其轨道相对于视线有明显倾斜的系外行星。作为答案，请说出你的方法所依据的物理现象。	1.0pt

C.2	让一颗质量为 m 的系外行星围绕一颗质量为 M 的恒星在半径为 R 的圆形轨道上公转，公转周期为 T，这颗行星的轨道平面与地球方向成一个角 θ。以你提出的方法探测这样一颗系外行星所需的相对频率测量精度。在你的答案中，用基本常数 R, T, θ, m 和 M 来表达你要测量频率的相对精度 $\Delta\omega/\omega$。	1.0pt

C.3	设系外行星和其恒星的质量分别等于地球和太阳的质量，圆轨道半径与地球到太阳的距离（$R \approx 1.5 \cdot 10^{11}$ m）相等，角度 $\theta = 60^\circ$。太阳的质量是地球质量的 33 万倍，地球绕太阳公转的周期为 1 年。求出一个整数 $\boxtimes$，使 10^{-n} 是你的方法所要求的相对频率精度。使用超短（飞秒）激光脉冲，在光波段（10^{15} Hz）频率测量精度约为 10 Hz。这个精度是否足以确认发现一颗系外行星？	0.25pt

激光技术 (10 points)

A.1 (1.0 pt)

$\tau =$

A.2 (0.25 pt)

$W_s =$

A.3 (0.25 pt)

$t_s =$

A.4 (0.5 pt)

$W_i =$

A.5 (0.25 pt)

$t_i =$

B.1 (0.5 pt)

$\Delta v =$

B.2 (2.0 pt)

$s =$

B.3 (0.5 pt)

B.4 (1.0 pt)

$s =$

B.5 (1.5 pt)

$W_c =$

C.1 (1.0 pt)

C.2 (1.0 pt)

$\Delta\omega/\omega =$

C.3 (0.25 pt)

$n =$

W3-2

W3-3

W3-4

W3-6

W3-7

W3-8

W3-9

W3-10

W3-11

W3-12

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CHN-S-1 T-3 W-13

W3-13

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W3-14

W3-15

CHN-S-3 E-1 C
Xiaoyu Ouyang

Experiment Q1

Cover sheet

Please return this cover sheet together with all the related question sheets.

Crystallography

Errors – no evaluation required. A list of equipment and recommendations for its use are at the very end of the problem.

Introduction

Crystallography is when a football stadium-sized synchrotron is used to measure distances on the order of several angstroms (the length of the peptide bond between amino acids 1.3 Å).

The main object studied in crystallography is a crystal. It can be thought of as one periodically repeating element in space - a unit cell. (Fig. 1)

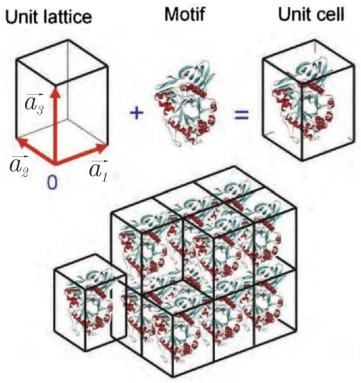


Figure 1: Unit cell represents whole crystal.

If $\rho(\vec{x})$ is some real function that describes the structure of the unit cell (for example, the positions of atoms or the density of electrons) and $\vec{a_1}, \vec{a_2}, \vec{a_3}$ – lattice vectors (may not be perpendicular to each other), then a complete crystal can be generated using the following equation:

$$\rho(\vec{x} + e \cdot \vec{a_1} + f \cdot \vec{a_2} + g \cdot \vec{a_3}) = \rho(\vec{x}), \qquad e, f, g, \in \mathbb{Z}. \tag{1}$$

In the case of 1D or 2D crystals, one ($\vec{a_1}$) or two vectors ($\vec{a_1}$ and $\vec{a_2}$), respectively, is sufficient to create a crystal from a unit cell.

The main task of the crystallographic experiment is to determine the crystal structure (in particular, $\rho(\vec{x}), \vec{a_1}, \vec{a_2}, \vec{a_3}$).

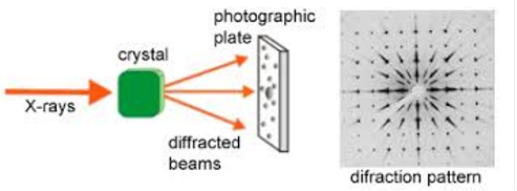


Figure 2: Scheme of a crystallographic experiment.

Due to the periodicity of the crystal, one method that can be used to determine the structure is radiation diffraction.

In the crystallographic experiment (Fig. 2), an incident beam with intensity I_0, wavelength λ and wave vector $\vec{k}_i$, ($|\vec{k}_i| = 2\pi/\lambda$) passes through the crystal (in the case of a 2D crystal $\vec{k}_i$, it is perpendicular to the crystal plane). The diffracted beam has the same wavelength λ, wave vector $\vec{k}_s$, $|\vec{k}_s| = |\vec{k}_i|$ and can be denoted by the scattering vector $\vec{q} = \vec{k}_s - \vec{k}_i$ (Fig. 3). It is clear that $q = 2k_i \sin(\theta/2)$. The angle θ between $\vec{k}_s$ и $\vec{k}_i$ can be considered much less than one, that is, $q << k_i$.

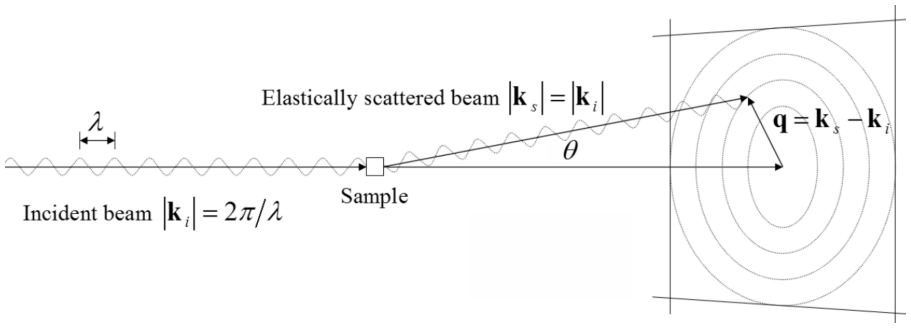


Figure 3: Scattering vector.

The complex amplitude $F(\vec{q})$ of the diffracted ray $\vec{q}$ is called the **structure factor** (since it is determined by the crystal structure) and is denoted $F(\vec{q})$. It has a module $|F|$ and φ phase:

$$F(\vec{q}) = |F| \cdot \exp(i\varphi). \tag{2}$$

The measured intensity $I(\vec{q})$ is the square of the modulus of the complex amplitude:

$$I = |F|^2 = FF^*. \tag{3}$$

Further processing of these intensities gives the densites $\rho(\vec{x})$. Usually the crystals under investigation contain small inorganic (salts or complex compounds) or even large organic molecules (proteins or DNA), and the density $\rho(\vec{x})$ determines the atomic positions of the compound. In this case, coherent X-rays with a wavelength of $1 \div 10$ Å(for example, synchrotron radiation) are used, corresponding to typical distances between atoms. If the distances are greater, a different wavelength should be chosen (for example, visible light can be used for optical gratings).

Now you will try to become a crystallographer. In Part A, using the example of a diffraction grating (1D crystal), we will study the basic laws. In Parts B, C, D you will define the lattice parameters, unit cell symmetries and then the detailed structure.

A. From slit to crystal

The simplest example of a crystal is an optical diffraction grating - it can be viewed as a one-dimensional crystal. Such a crystal has a slit (width b) and an inter-slit gap as a unit cell (Fig. 4A). The lattice period $a_1 = a$ is a one-dimensional vector that generates the crystal.

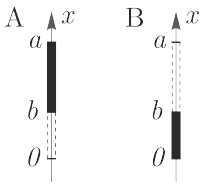


Figure 4: Unit cells (A and B) of a 1D crystal (diffraction grating). The non-transparent areas are shown in black.

When observing Fraunhofer diffraction on a diffraction grating, the light intensity I depends on the light propagation angle θ as follows

$$I(\theta) = \frac{I_0}{N^2}\left(\frac{\sin(\frac{N\pi a \sin\theta}{\lambda})}{\sin(\frac{\pi a \sin\theta}{\lambda})}\right)^2 \cdot \left(\frac{\sin(\frac{\pi b \sin\theta}{\lambda})}{(\frac{\pi b \sin\theta}{\lambda})}\right)^2, \tag{4}$$

where I_0 is the intensity at $\theta = 0$, N is the number of illuminated slits of the diffraction grating.

To understand equation 4, you need to know that:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1. \tag{5}$$

A.1	Write the formula 4 using the scattering vector q ($q \ll k_i$).	0.3pt
A.2	Find scattering vector q for the maximum numbered h for a diffraction grating with a period a.	0.2pt
A.3	Let q_1 be the scattering vector for the first maximum. Express q in terms of q_1 for intensity maxima. How are q_1 and a related?	0.2pt
A.4	Observe the diffraction of samples DG1-DG5. Determine experimentally q_1, and a for each sample. Draw a scheme of your setup, write the quantities you measure, and write down the formulas for the calculations.	1.0pt
A.5	Conduct an experiment and determine the a/b ratio for samples DG3, DG4, DG5. Explain your solution using formulas, diagrams, and pictures. It is known that $b \leq a/2$.	1.5pt

It can be noted that the period of the crystal $a_1 = a$ is responsible for the periodicity of the diffraction maxima, and the unit cell parameters (slit width b) are responsible for the intensity of the maxima. This fact is used to simplify the calculation of the relative intensities of the maxima by using the structure factor $F(q)$:

$$F(q) \sim \int \rho(x)\exp(iqx)dx, \tag{6}$$

where $\rho(x)$ – is transmission (the ratio of the amplitudes of the transmitted wave to the incident wave), q – the scattering vector, which describes the position in the diffraction pattern. Integration is carried out

over the entire unit cell. If we replace q with the position of this or that maximum, it will be possible to find the relative intensities of the maxima as $I = |F|^2$. Here and below $\rho(x) \in \mathbb{R}$.

The positions of the intensity maxima for crystal diffraction (called **reflexes**) can always be represented as a sum of vectors (called **reciprocal lattice vectors**):

1-D: $\vec{q} = h \cdot \vec{q}_1$, where $h \in \mathbb{Z}$;

2-D: $\vec{q} = h \cdot \vec{q}_1 + k \cdot \vec{q}_2$, where $h, k \in \mathbb{Z}$;

3-D: $\vec{q} = h \cdot \vec{q}_1 + k \cdot \vec{q}_2 + l \cdot \vec{q}_3$, where $h, k, l \in \mathbb{Z}$.

Thus, each reflex for a 3D crystal can be denoted by three numbers (h, k, l) (and (h) for 1D and (h, k) for 2D) and has its own complex amplitude $F(h, k, l)$ and intensity $I(h, k, l)$.

A.6	Write down $\rho(x)$ for the unit cell of the diffraction grating from A1 (Fig. 4A). Use the coordinate system as shown in the figure. Suppose that the unit cell is such that the period of the lattice a is p times the width of the slit b: $a = pb, \quad p \in \mathbb{N}$. Calculate the structure factor $F_A(h)$ for this unit cell for reflex h. Record your answer using h and q_1. What maxima have intensity 0? Write the equation for h for such maxima.	0.7pt

A.7	Consider another unit cell (Fig. 4B) of the diffraction grating. Calculate the structure factor $F_B(h)$ for this unit cell for reflex h. What reflexes of this diffraction grating have an intensity of 0? Write the equation for h for such reflexes.	0.7pt

A.8	These two diffraction gratings described above are illuminated with light of the same intensity. Find the quotients $I_{A,h=0}/I_{B,h=0}$ and $I_{A,h=1}/I_{B,h=1}$.	0.4pt

B. 2D crystal

In this part you will go through the first step in defining the structure – finding the lattice parameters.

The reciprocal lattice vectors in the case of 2D

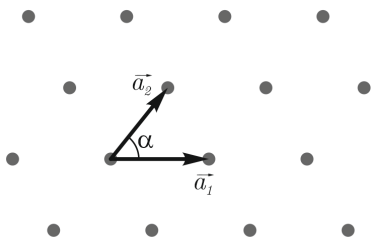


Figure 5: 2D crystal with vectors $\vec{a}_1$ and $\vec{a}_2$ as lattice vectors. The points represent the positions of the atoms in this crystal.

Let us consider diffraction on a two-dimensional crystal. A schematic 2D crystal of point atoms is shown in Fig. 5. Lattice vectors $\vec{a}_1$ and $\vec{a}_2$ has an angle $\alpha \leq 90^\circ$ between them.

If the plane of a given 2D crystal is placed perpendicular to the beam, then a periodic diffraction pattern appears on the screen behind the crystal, the position of the maxima of which is described by the expression $\vec{q} = h \cdot \vec{q}_1 + k \cdot \vec{q}_2$, where $h, k \in \mathbb{Z}$.

You can get an idea of such a diffraction pattern by looking at the diffraction patterns of UC1-UC7 samples.

B.1	Find the angle β between the vectors $\vec{q}_1$ and $\vec{q}_2$ and their lengths q_1, q_2. Please note that these vectors must be of minimum length and the angle between them must be $\leq 90°$. Express your answer through the crystal parameters a_1, a_2, α (Fig. 5)	1.0pt

Samples UC1-UC4: the simplest 2D crystals

Consider four 2D crystals A, B, C, D (Fig. 6). These crystals have a square unit cell with perpendicular lattice vectors of equal length. The unit cell of each crystal contains an atom (a non-transparent square with a side b on a transparent film). Unit cells C and D also contain displaced atoms of the second type. In crystal C, the size of an atom of the second type b' is larger than that of the first: $b' > b$. In crystal D, the second atom is the same size as the first, and is displaced half a period to the right and half a period up.

B.2	For crystals A and D, find the complex amplitude modulus $\lvert F(h,k)\rvert$ for the reflex (h,k). Express your answer in terms of a (crystal period) and b (atom size). It is enough to indicate an expression that is true for all reflexes except for the central one ($h = 0$, $k = 0$)	1.0pt

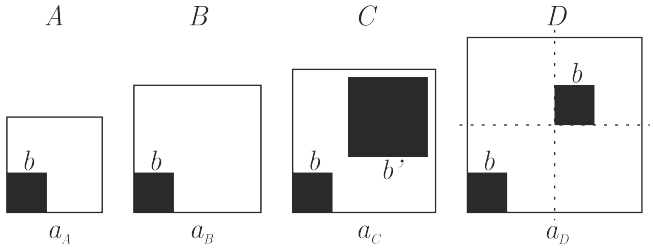


Figure 6: The unit cells for crystals A-D are square. It is known that $a_B > a_A$. No other size ratios are known.

B.3	Look at the diffraction patterns of samples UC1-UC4. For each UC1-UC4 sample, experimentally determine the crystal lattice period a_{UC1}, a_{UC2}, a_{UC3}, a_{UC4}.	0.6pt

B.4	For each UC1-UC4 sample, find the corresponding crystal structure among Fig. 6. Explain your choice using diagrams, pictures and formulas.	0.4pt

B.5	Determine the size of the atom b.	0.8pt

Samples UC5-UC6: Still Simple 2D Crystals

B.6	Observe the diffraction patterns of samples UC5, UC6, UC7. Determine experimentally the parameters a_1, a_2 and the angle α for each sample. Explain which parameters of the diffraction pattern you are using with the help of diagrams and figures.	1.2pt

C. Crystal Symmetries

The second step after calculating the parameters of the unit cell is to determine the symmetry of the unit cell.

Symmetries: Understand Theoretically

The unit cells of real crystals often contain several molecules with some symmetry between them (Fig. 7). Knowing these symmetries greatly simplifies the process of defining the structure. The symmetry of the unit cell causes reciprocal symmetries and systematic absences (reflexes with zero intensity for each unit cell with this symmetry). Systematic absence is determined by special conditions for h and k (Fig. 8).

Typical Symmetries for Reflex Intensities:

- **mirror symmetry** about some straight line. This line is called the axis of symmetry and is denoted by the equation of this line;
- **rotational symmetry** of order m (denoted as C_m for a specific center of rotation, $m \in \mathbb{N}$). When rotated through an angle of $360^\circ \cdot n/m$, $n \in \mathbb{Z}$, the picture transforms into itself).

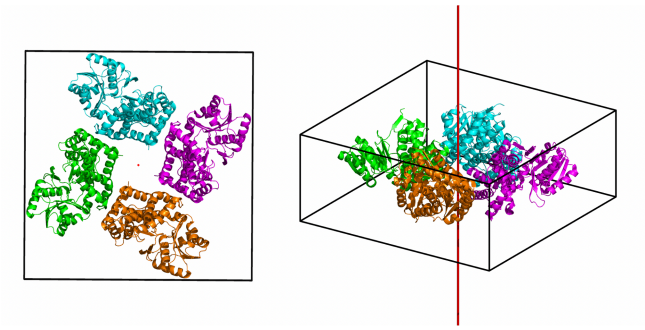


Figure 7: This unit cell has the rotation symmetry C_4: rotation around the red axis by $n \cdot 90^\circ$ degrees, $n \in \mathbb{Z}$ gives the same cell.

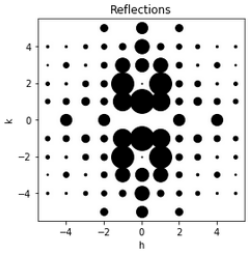


Figure 8: Reflexes $(h = 2n+1, k = 0)$ are systematically absent. Note: Reflex $(0, 0)$ with relatively high intensity is omitted here for clarity.

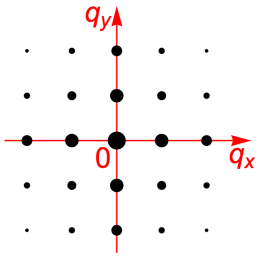


Figure 9: Diffraction image of 2D crystal.

Consider a diffraction image (Fig. 9) from a 2D crystal with perpendicular lattice vectors equal to each other ($a_1 = a_2$). This diffraction image (only reflexes with $|h|, |k| \leq 2$) is shown in the figure.

C.1	Specify h and k for the point that will be the center of rotation. What orders of rotational symmetry m are possible for a given image? Draw all possible axes of mirror symmetry in the image. Name your lines.	0.3pt

The axis of symmetry is – a straight line, which means it can be indicated as an equation of the straight line $c_1 \cdot q_x + c_2 \cdot q_y = d$, where c_1, c_2, d are some coefficients.

C.2	Specify the equation of the straight line for each axis of mirror symmetry drawn in the previous task. Do not forget to note which equation corresponds to which line.	0.2pt

C.3	For each rotation symmetry and axis of mirror symmetry, write down the corresponding notation (C_m for rotation and equation for mirror symmetry) and the equation for the intensities $I(q_x, q_y)$, which should take place if this symmetry element is present.	0.4pt

C.4	Write down the equation for the intensities of the reflexes (h, k) and $(-h, -k)$. What symmetry from question C1 corresponds to this equation? Explain your answer.	0.2pt

Let us consider some crystals with unit cells as in Fig. 10. Black squares represent non-transparent elements, white squares represent transparent elements. The unit cell of crystal 2 was obtained by mirror symmetry of the initial unit cell relative to the $x = 0$ axis. The unit cell of crystal 3 was obtained by the

mirror symmetry of the initial unit cell about the $y = x$ axis. Crystal 4 is obtained by shifting the original one by the vector (x_1, y_1).

C.5	Using the definition of the structure factor and symmetry find the structural factors $f_2(q_x, q_y)$, $f_3(q_x, q_y)$, $f_4(q_x, q_y)$ for crystals 2, 3, 4, respectively. Express your answer in terms of the structure factor $F(q_x, q_y) = f_1(q_x, q_y)$ of crystal 1.	0.4pt

C.6	Consider an arbitrary 2D crystal (Fig. 5). Indicate what orders of m symmetry of rotation can be in 2D crystals. Explain the answer.	0.5pt

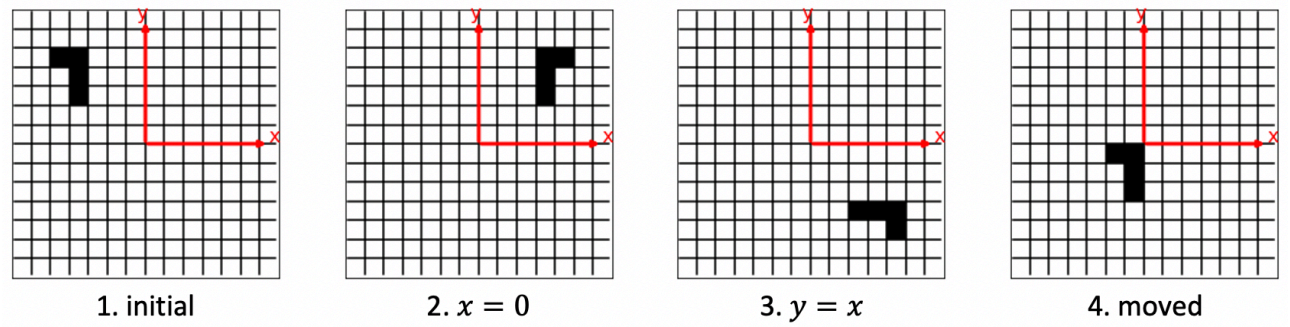


Figure 10: The unit cells of crystals 2, 3, 4 were obtained from 1 using different symmetries.

Symmetries: Observe Experimentally

The answer sheet contains unit cells for the patterns **PG1-PG9**. Having studied the symmetry of unit cells and diffraction patterns, and reflex absences, we will find a correspondence between them.

C.7	Determine what symmetries the crystals with unit cells K, L, M, N, and P, Q, R, S, T have. (Fig. in the answer sheet). Draw the axes of mirror symmetry, at the bottom of the picture indicate which rotational symmetries are present on it.	0.9pt

C.8	Observe the diffraction patterns of the samples **PG 1, 2, 5, 8**. These samples correspond to the unit cells K, L, M, N. Determine what symmetries the given diffraction patterns have. Find the correspondence between patterns and unit cells.	0.8pt

The symmetries of diffraction patterns can be calculated using the sum rule (the complex amplitude of a complex object is the sum of the complex amplitudes of the parts of that object) and knowledge of how the symmetries change the complex amplitudes of the terms. In addition, the sum rule allows you to define absences.

C.9	Observe the diffraction of the samples **PG 3, 4, 6, 7, 9**. These samples correspond to the unit cells P, Q, R, S, T. Find the correspondence between samples and unit cells. Explain your solution using formulas, diagrams and pictures.	1.0pt

C.10	Observe the diffraction pattern of the UC8 sample. Could this sample be a crystal? Explain your answer.	0.3pt

D. You still need phases ...

When you know the parameters of the unit cell and the symmetry of the unit cell, it is time for the last step - to determine the complete structure of the crystal.

Phase problem ...

When radiation is scattered by a crystal, the complex amplitudes can be calculated using the formula (the so-called Fourier transform)

$$F(\vec{q}) \sim \int \rho(\vec{x}) \cdot \exp(i\vec{q}\vec{x}) d\vec{x}. \tag{7}$$

To determine the density of a crystal using a diffraction image, it is necessary to perform the inverse Fourier transform:

$$\rho(\vec{x}) \sim \int F(\vec{q}) \cdot \exp(-i\vec{q}\vec{x}) d\vec{q}. \tag{8}$$

Since we have discrete reflexes, the above formula can be written as follows

$$\rho(\vec{x}) \sim \sum |F(\vec{q})| e^{i\varphi} \cdot \exp(-i\vec{q}\vec{x}), \tag{9}$$

where summation is performed over all reflexes. In fact, it is enough to take only the brightest reflexes, since their contribution to the sum is the largest.

The experimentally measured intensities $I(\vec{q})$ make it possible to determine only the modulus $|F(\vec{q})|$. Unfortunately, the intensities do not contain information about the phases φ, so it is impossible to directly calculate $\rho(\vec{x})$ from the intensities. This problem is called **phase problem**.

The usual way to solve the phase problem is to get some approximate phases, then calculate the density of the crystals using them, then update the phases using that density, and repeat this until you are satisfied with the resulting structure.

There are several methods for obtaining initial phases. One assumes that you already have a known crystal structure, and that the known and unknown crystals are structurally similar. In this case, you can simply use the phases calculated from the known structure with the measured intensities in the inverse Fourier transform (9) for the unknown structure.

... which you successfully solve

You have 3 different 2D crystals: MR0, MR1, MR2. The unit cell of each of them is a 4x4 square, with some squares non-transparent ($\rho = 0$), and others transparent ($\rho = 1$). The unit cell MR0 is known (Fig. 11). The structures MR1 (possible variants are shown in Fig. 12) and MR2 (contains 7 white squares) are unknown, but you know that they are quite similar to MR0.

Using the above approach, you will define the structures MR1 and MR2. Reflexes phases (in radians) with $|h|, |k| \leq 2$ were preliminary calculated (Fig. 13) for the MR0 structure (in the coordinate system shown in Fig. 11). Hint: the imaginary parts of the densities $\mathrm{Im}\rho(\vec{x})$ vanish with good accuracy, which means that it is enough to take into account only the real part of each term (9).

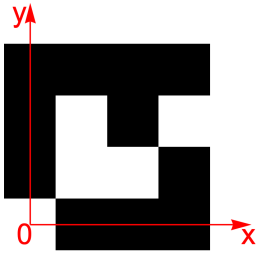


Figure 11: MR0 structure with coordinate system. White squares – transparent, black – non-transparent.

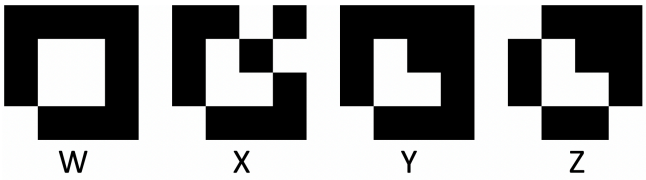


Figure 12: Possible unit cells of the MR1 crystal.

k \ h	-2	-1	0	1	2
2	3.142	0.4636	0	-0.4636	3.142
1	0	-0.4636	2.034	-1.571	0
0	3.142	-1.571	0	1.571	3.142
-1	0	1.571	-2.034	0.4636	0
-2	3.142	0.4636	0	-0.4636	3.142

Figure 13: Reflexes phases (in radians) for the MR0 structure for the $|h| \leq 2, \quad |k| \leq 2$ reflexes.

D.1	The crystal (MR0 or MR2) is illuminated with light with an intensity of I_0. Find the intensity of the maximum at $\vec{q} = 0$.	1.0pt
D.2	Determine the structure of the unit cell of the MR1 crystal. The MR1 crystal has one of the indicated units cell (Fig. 12). Describe your solution.	2.0pt
D.3	Determine the unit cell structure of the MR2 crystal. The structure of MR2 is similar to MR0: two non-transparent squares have become transparent.	2.0pt

Equipment

Equipment list

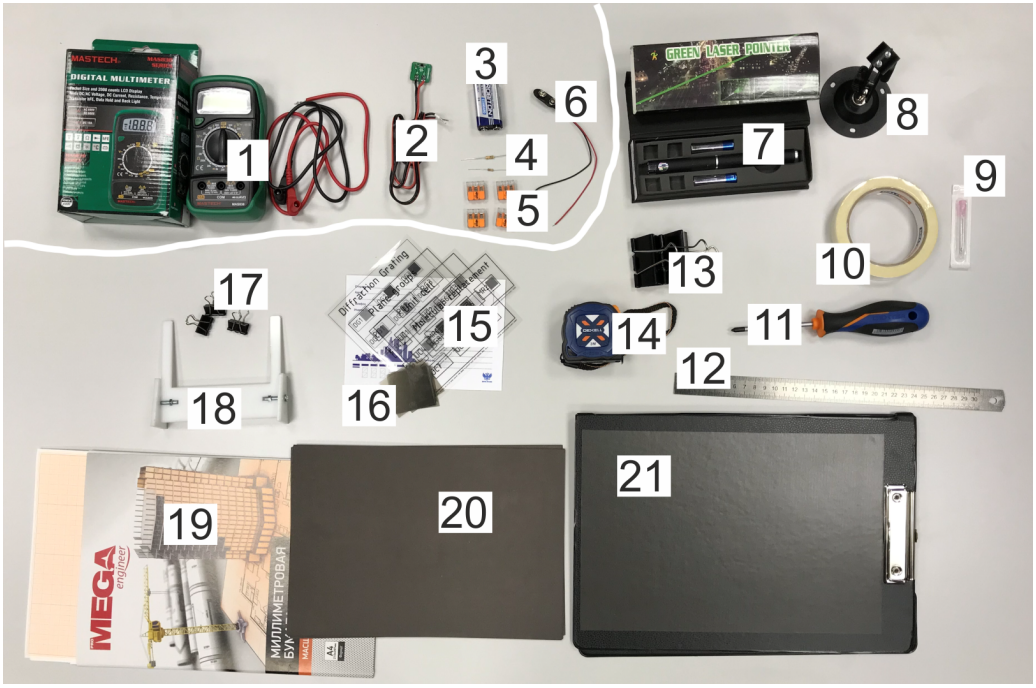


Figure 14: Equipment. In the upper left corner – details of the intensity detector.

1. Multimeter
2. Photodiode with magnet
3. Battery
4. Resistors 10 kOhm and 200 kOhm
5. Splicing Connectors (4 pcs.)
6. Battery holder
7. Red laser (wavelength λ = 630 nm)
8. Laser stand
9. Needle
10. Masking tape
11. Screwdriver
12. Ruler
13. Large clips (2 pcs.)
14. Tape Measure
15. Slides ("Diffraction grating", "Unit cell", "Plane group", "Molecular replacement") with samples

16. Film for reducing laser intensity
17. Small clips (4 pcs.)
18. Slide holder (needs to be assembled)
19. Graph paper
20. Cardboard
21. Tablet with magnetic layer

Samples

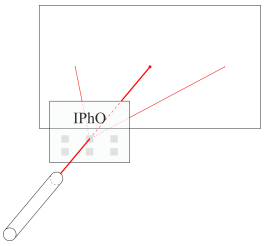


Figure 15: Place and illuminate the samples as shown.

Samples should be illuminated with a laser as follows. Position the sample so that the caption (slide title in English) is on top of the slide and reads from left to right. In this case, the laser should be in front of the slide, and the screen for observing should be behind the slide (Fig. 15).

Intensity detector

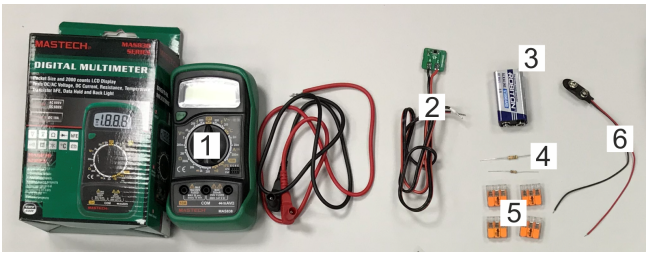


Figure 16: Items of the intensity detector.

To measure intensities, use a photodiode connected to a voltmeter, resistor and battery according to the circuit shown in Fig. 17. In such a circuit, the current through the diode is proportional to the intensity of the incident light.

Pay attention to the polarity of the connection. On the battery block, the red wire is positive. By default, use a 10 kΩ resistor, in case the measured intensities are very small – use a 200 kΩ resistor.

To use the connectors (fig. 18), do the following. Lift the lever up, insert the wire into the connector, lower the lever. All three connector points are electrically connected to each other.

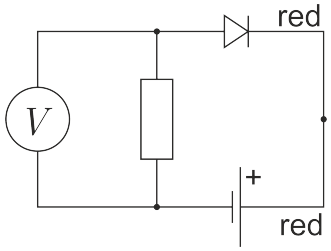


Figure 17: Intensity detector circuit. Pay attention to the polarity of the connection.



Figure 18: Scheme of splicing connector. All three connector points are electrically connected to each other.

Laser

Mount the laser in the stand 8. Use a large clip to fix the laser on. If you need to make the laser beam narrower, then seal the laser hole with masking tape and make a hole in it with a needle. If you want to reduce the intensity of the laser – use a darkening film. Figure 19 is for your reference about that.

!WARNING! Direct exposure to the laser beam is hazardous to the eyes. Do not point the laser beam at yourself or other people. If you don't use the laser – turn it off.

!WARNING! A sharp needle is dangerous: you can get hurt! Use it carefully. When not in use, put the needle in the protective cover.



Figure 19: Laser is kept switched on using large clip. To reduce beam size use sticky tape and a needle. Darkening film in a holder (large clip) to reduce laser intensity.

晶体学

误差不需要计算。问题下方有一份设备清单和使用建议。

引言

晶体学是指用一个足球场大小的同步加速器来测量 以埃为单位的距离（氨基酸之间的肽键长度为 1.3Å）

晶体学研究的主要对象是晶体。它可以被看作是空间中一个周期性重复的单元–原胞。(图 1)

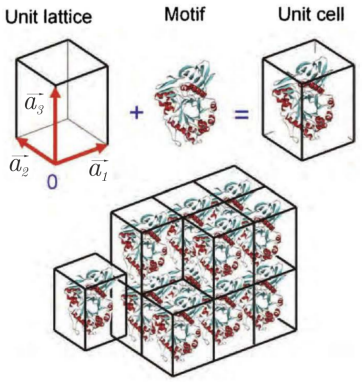


图 1：用原胞代表整个晶体。

如果 $\rho(\vec{x})$ 是一些描述原胞结构的实函数 (例如，原子的位置或电子的密度)，而 $\vec{a}_1, \vec{a}_2, \vec{a}_3$ 是晶格矢量 (可以不互相垂直)，那么一个完整的晶体可以用下面的公式生成。

$$\rho(\vec{x} + e \cdot \vec{a}_1 + f \cdot \vec{a}_2 + g \cdot \vec{a}_3) = \rho(\vec{x}), \qquad e, f, g, \in \mathbb{Z}. \tag{1}$$

在一维或二维晶体的情况下，分别用一个（$\vec{a}_1$）或两个矢量（$\vec{a}_1$ 和 $\vec{a}_2$），就足以从一个原胞中产生一个晶体。

晶体学实验的主要任务是确定晶体结构 (特别是 $\rho(\vec{x}), \vec{a}_1, \vec{a}_2, \vec{a}_3$)。

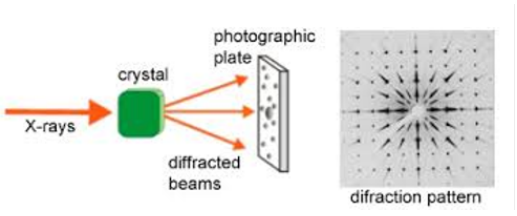


图 2：晶体学实验方案

由于晶体的周期性，可以用来确定结构的一种方法是衍射。

在晶体学实验中 (图 2)，强度为 I_0、波长为 λ、波矢为 $\vec{k}_i$ 的入射光束 ($|\vec{k}_i| = 2\pi/\lambda$) 穿过晶体 (在二维晶体的情况下 $\vec{k}_i$ 垂直于晶体平面)。衍射光束具有相同的波长λ，波矢 [illegible] ($|\vec{k}_s| = |\vec{k}_i|$)，可以用散射矢量 $\vec{q} = \vec{k}_s - \vec{k}_i$ 表示（图 3)。显然，$q = 2k_i \sin(\theta/2)$。$\vec{k}_s$ 和 $\vec{k}_i$ 之间的角度 [illegible] 可以认为远小于 1，即 $q \ll k_i$。

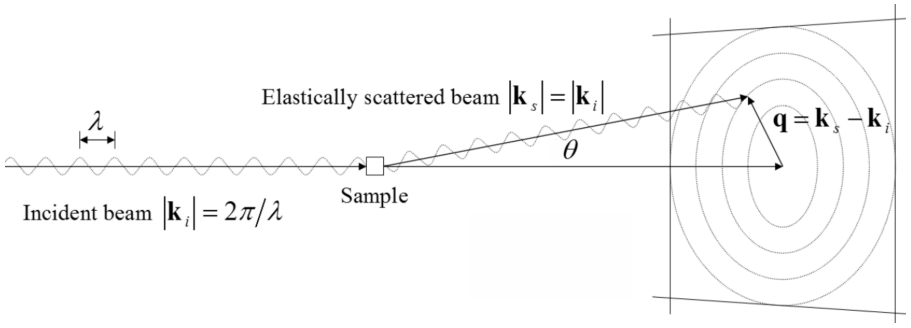


图 3：散射矢量

衍射射线的复振幅 $F(\vec{q})$ 称为结构因子 (因为它是由晶体结构决定的)，并表示为 $F(\vec{q})$。它有一个模 $|F|$ 和相位 φ。

$$F(\vec{q}) = |F| \cdot \exp(i\varphi). \tag{2}$$

测量到的强度 $I(\vec{q})$ 是复振幅的模平方。

$$I = |F|^2 = FF^*. \tag{3}$$

对这些强度进一步处理后，可得到密度 $\rho(\vec{x})$。通常，被调查的晶体含有小的无机物 (盐类或复杂的化合物)，甚至是大的有机分子 (蛋白质或 DNA)，密度 $\rho(\vec{x})$ 决定了化合物的原子位置。在这种情况下，使用波长为 1-10 埃的相干 X 射线（例如，同步辐射），对应于原子之间的典型距离。如果距离更大，则应选择不同的波长（例如，可见光可用于光学光栅）。

现在你将尝试成为一名晶体学家。在 A 部分，以衍射光栅（1D 晶体）为例，我们将研究基本规律。在 B、C、D 三部分中，你将确定晶格参数、原胞对称性，然后是详细的结构。

A. 从狭缝到晶体

晶体最简单的例子是光学衍射光栅–它可以看作是一维晶体。这样的晶体有一个狭缝 (宽度 ☒) 和一个狭缝间的间隙，构成原胞 (图 4A)。晶格周期 $a_1 = a$ 是生成晶体的一维向量。

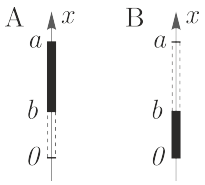


图 4：一维晶体（衍射光栅）的原胞（A 和 B）。非透明区域用黑色显示。

当在衍射光栅上观察夫琅禾费衍射时，光强 I 取决于光的传播角 θ，如下所示

$$I(\theta) = \frac{I_0}{N^2}\left(\frac{\sin(\frac{N\pi a \sin\theta}{\lambda})}{\sin(\frac{\pi a \sin\theta}{\lambda})}\right)^2 \cdot \left(\frac{\sin(\frac{\pi b \sin\theta}{\lambda})}{(\frac{\pi b \sin\theta}{\lambda})}\right)^2, \tag{4}$$

其中，I_0 为 $\theta = 0$ 时的强度，N 为衍射光栅的被照明狭缝数。

要理解方程 4，你需要知道：

$$\lim_{x\to 0}\frac{\sin x}{x}=1. \tag{5}$$

A.1	用散射矢量 q（$q \ll k_i$）写出公式 4。	0.3pt
A.2	对于最大编号为 h 的周期为 a 的衍射光栅，找出散射矢量 q。	0.2pt
A.3	设 q_1 是第一个极大值对应的散射向量。请用 q_1 来表示强度最大值对应的 ☒，☒1 和 ☒ 的关系如何?	0.2pt
A.4	观察样品 DG1-DG5 的衍射情况。通过实验确定每个样品的 q_1 和 a。画出你的实验方案，写出你测量的量，并写出计算公式。	1.0pt
A.5	进行实验并确定样品 DG3、DG4、DG5 的 a/b 比率。请用公式、图表和图片解释你的解决方案。已知 $b \leq a/2$。	1.5pt

可以注意到，晶体的周期（$a_1 = a$）负责衍射最大值的周期性，而原胞参数（狭缝宽度 ☒）负责最大值的强度。这个事实可以用结构因子 $F(q)$ 来简化计算最大值的相对强度。

$$F(q) \sim \int \rho(x)\exp(iqx)dx, \tag{6}$$

其中 $\rho(x)$–是透射 (透射波与入射波的振幅之比)，q–是散射矢量，它描述了衍射图案中的位置。积分是在整个原胞中进行的。如果我们用这个或那个最大值的位置来代替 ☒，就可以找到最大值的相对强度为 $I = |F|^2$。这里以 $\rho(x) \in \mathbb{R}$. 表示。

晶体衍射的强度最大值的位置（称为反射（reflexes））总是可以用矢量的和来表示（称为倒格矢）。

1-d: $\vec{q} = h \cdot \vec{q}_1$, где $h \in \mathbb{Z}$;

2-d: $\vec{q} = h \cdot \vec{q}_1 + k \cdot \vec{q}_2$, где $h, k \in \mathbb{Z}$;

3-d: $\vec{q} = h \cdot \vec{q}_1 + k \cdot \vec{q}_2 + l \cdot \vec{q}_3$, где $h, k, l \in \mathbb{Z}$.

因此，三维晶体的每一个反射（reflex）都可以用三个数字来表示 (☒,☒,☒)((h) 表示 1D，(h, k) 表示 2D)，并有自己的复振幅 $F(h, k, l)$ 和强度 $I(h, k, l)$

A.6	从 A1 开始写下 $\rho(x)$ 为原胞的衍射光栅 (图 4A)。使用如图所示的坐标系。假设原胞是这样的，格子的周期 ☒ 为 ☒ 乘以狭缝宽度 ☒：$a = pb, \quad p \in \mathbb{N}$。计算该原胞的反射 h 的结构系数 $F_A(h)$。用 ☒ 和 ☒1 记录你的答案。哪些主极大的强度为 0？写出此类主极大对于 h 的方程。	0.7pt
A.7	考虑另一个衍射光栅的单元（图 4B)。计算这个单元格为反射 h 的结构因子 $F_B(h)$。该衍射光栅的什么反射强度为 0？写出反射 h 的方程。	0.7pt

A.8	上述这两个衍射光栅被相同强度的光照亮。求商：$I_{A,h=0}/I_{B,h=0}$，$I_{A,h=1}/I_{B,h=1}$	0.4pt

B. 2D 晶体

在这一部分中，您将经历定义结构的第一步–寻找晶格参数。

2D 倒格矢

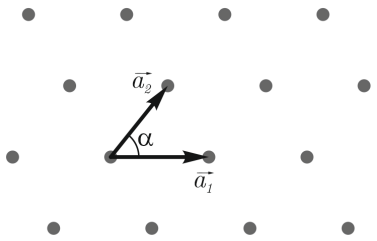


图 5：以矢量 $\vec{a_1}$ 和 $\vec{a_2}$ 为晶格矢量的二维晶体。这些点代表了该晶体中原子的位置。

让我们考虑二维晶体上的衍射。图 5 为点原子的二维晶体示意图，晶格向量 $\vec{a_1}$ 和 $\vec{a_2}$ 之间的角度：⍰⩽90⍰。

如果将给定的二维晶体的平面垂直于光束放置，那么在晶体后面的屏幕上就会出现周期性的衍射图案，其最大值的位置表达式为 $\vec{q} = h \cdot \vec{q}_1 + k \cdot \vec{q}_2$，其中 $h, k \in \mathbb{Z}$。

你可以通过观察 UC1-UC7 样品的衍射图案来了解这样的衍射图案。

B.1	找出向量 $\vec{q}_1$ 和 $\vec{q}_2$ 之间的角度和它们的长度 ⍰1、q_2。请注意，这些向量的长度必须是最小的，并且它们之间的角度必须 $\leq 90°$。通过晶体参数 a_1, a_2, α 来表达你的答案 (图 5)	1.0pt

样品 UC1-UC4：最简单的 2D 晶体

观察四个 2D 晶体 A、B、C、D（图 6）。这些晶体有一个正方形的原胞，其垂直的晶格向量长度相等。每个晶体的原胞中都包含一个原子 (在透明薄膜上有一个边长为 ⍰ 的非透明正方形)。单元格 C 和 D 也包含第二种类型的位移原子。在晶体 C 中，第二种类型的原子 b' 的尺寸比第一种类型的原子大：$b' > b$。在晶体 D 中，第二种原子与第一种原子大小相同，向右移动半个周期，向上移动半个周期。

B.2	对于晶体 A 和 D，请求出反射 (h, k) 的复振幅的模 $\lvert F(h, k)\rvert$。用 a(晶体周期) 和 b(原子大小) 来表示答案。给出一个除中心反射（$h = 0$，$k = 0$）外，其他反射都成立的表达式即可。	1.0pt

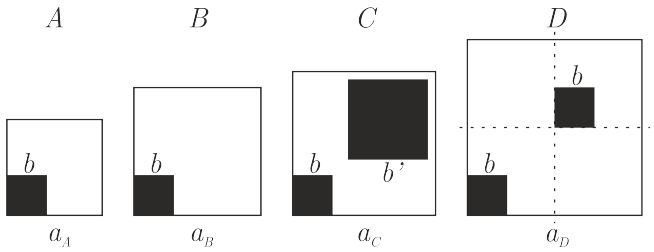


图 6：晶体 A-D 的原胞为正方形。已知 $a_B > a_A$。其他尺寸比未知。

B.3	观察样品 UC1-UC4 的衍射图案。对于每个 UC1-UC4 样品，请用实验方法确定晶体的晶格周期 $a_{UC1}, a_{UC2}, a_{UC3}, a_{UC4}$.	0.6pt
B.4	对于每一个 UC1-UC4 样品，请在图 6 中找到相应的晶体结构，用图示、图片和公式解释你的选择。	0.4pt
B.5	确定 b 原子的大小	0.8pt

样品 UC5-UC6：仍然是简单的 2D 晶体

B.6	观察样品 UC5、UC6、UC7 的衍射图案。通过实验确定每个样品的参数 a_1、a_2 和角度 α。用图表解释你使用了衍射图样的哪些参数。	1.2pt

C. 晶体对称性

计算出原胞的参数后，第二步是确定原胞的对称性。

对称性。理论理解。

真实晶体的原胞中往往包含多个分子，它们之间具有一定的对称性（图 7）。了解这些对称性可以大大简化结构的确定过程。原胞的对称性会引起倒格矢的对称性和系统性的缺级（具有这种对称性的原胞的某些反射级的强度为零）。系统性缺级是由 h 和 k 的特殊条件决定的（图 8）。

反射强度的典型对称性

- 关于某条直线的镜像对称。这条直线称为对称轴，用这条直线的方程表示。
- 对于特定的旋转中心，序数为 ☒ 的旋转对称（表示为 C_m），$m \in \mathbb{N}$。当旋转角度为 $360^\circ \cdot n/m$ 时，$n \in \mathbb{Z}$，图像会变换回自己）。)

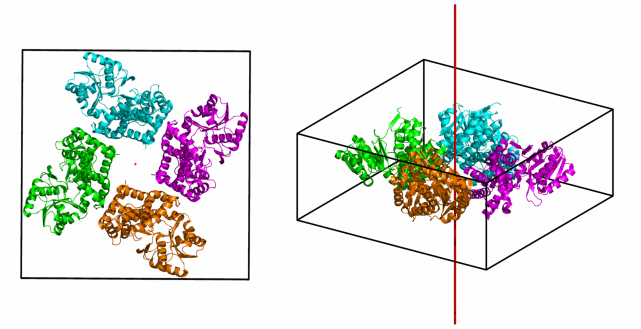


图 7：该单元格具有旋转对称性 C_4：绕红轴旋转 $n \cdot 90^\circ$ 度，$n \in \mathbb{Z}$，将得到相同的单元格。

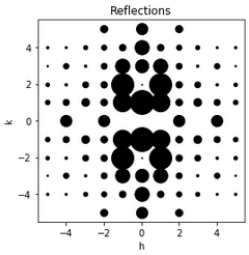


图 8：反射 $(h = 2n + 1, k = 0)$ 缺级。

注：为了清楚起见，这里省略了强度相对较高的反射（0,0)。

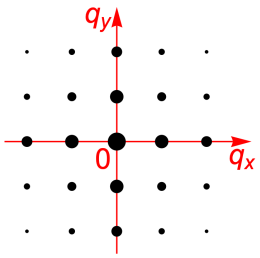


图 9：2D 晶体的衍射图像。

观察一个二维晶体的衍射图像 (图 9)，其垂直的晶格矢量彼此相等 $a_1 = a_2$。此衍射图象 ($|h|, |k| \leq 2$ 情况下的唯一反射) 如图所示。

C.1	对于旋转中心点，指出 ☒ 和 ☒。对于给定的图像，哪些阶数 ☒ 的旋转对称是可能的？在图像中画出所有可能的镜像对称轴。自行命名对称轴。	0.3pt

对称轴是直线，可以表示直线的方程为 $c_1 \cdot q_x + c_2 \cdot q_y = d,$，其中 ☒1，☒2，$d$ ～是一些系数。

C.2	指出上一个任务中画出的每条镜面对称轴的直线方程。不要忘记标出哪个方程对应哪条直线。	0.2pt

C.3	对于每个旋转对称和镜面对称的轴，写下相应的旋转序数（C_m）和镜面对称的方程，以及强度 $I(q_x, q_y)$ 的方程，如果存在这个对称元素，应该发生。	0.4pt

C.4	对于反射（□,□）和（-□,-□），写下反射强度的方程。C1 中的什么对称性与这个等式相对应？请解释你的答案。	0.2pt

让我们考虑一些具有如图 10 中的原胞的晶体。黑色方块代表非透明元素，白色方块代表透明元素。晶体 2 的原胞是通过初始原胞相对于 □=0 轴的镜面对称得到的。晶体 3 的原胞是通过初始原胞关于 □=□ 轴的镜面对称得到的。晶体 4 是由最初的一个原胞移动矢量 (x_1，y_1) 得到的。

C.5	利用结构因子的定义和对称性分别找出晶体 2、3、4 的结构因子 $f_2(q_x, q_y)$、$f_3(q_x, q_y)$、$f_4(q_x, q_y)$。用晶体 1 的结构系数 $F(q_x, q_y) = f_1(q_x, q_y)$ 表示答案。	0.4pt

C.6	考虑一个任意的 2D 晶体（图 5）。指出在 2D 晶体中，旋转的对称性可以是什么阶数的 m。请解释答案。	0.5pt

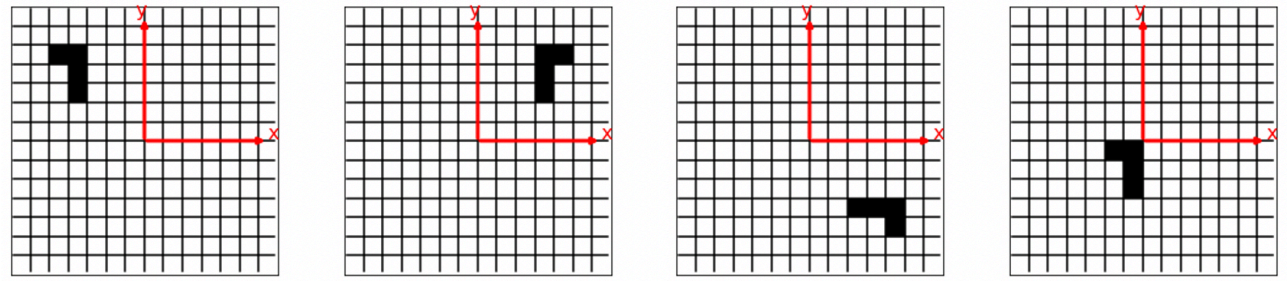


图 10：利用不同的对称性从 1 中得到晶体 2、3、4 的原胞。

对称性。实验观察

答卷中包含了图案 PG1-PG9 的原胞。在研究了原胞和衍射图案的对称性以及反射缺级之后，我们将发现它们之间的对应关系。

C.7	确定具有原胞的晶体 □、□、□、□ 和 □、□、□、□、□ 有哪些对称性。(答卷中的图)。画出镜面对称的轴线，在图的下方注明其上有哪些旋转对称。	0.9pt

C.8	观察样品 PG 1、2、5、8 的衍射图案。这些样品对应于单元格 □、□、□、□。确定给定的衍射图案有哪些对称性。找出图案和原胞之间的对应关系。	0.8pt

衍射图案的对称性可以使用求和规则（复杂物体的复振幅是该物体各部分的复振幅之和）和对称性如何改变复振幅的知识来计算。此外，求和规则还允许你确定缺级。

C.9	观察样品 PG 3、4、6、7、9 的衍射。这些样品对应于原胞 ☒、☒、☒、☒、☒。找出样品和原胞之间的对应关系。用公式、图表和图片来解释你的答案。	1.0pt

C.10	观察 UC8 样品的衍射图案。该样品会不会是晶体？请解释你的答案。	0.3pt

D. 无论如何，你都会需要相位...。

当你知道了原胞的参数和对称性后，就到了最后一步–确定晶体的完整结构。

Фазовая проблема...

当辐射被晶体散射时，可以用公式（所谓的傅里叶变换）计算出复振幅。

$$F(\vec{q}) \sim \int \rho(\vec{x}) \cdot \exp(i\vec{q}\vec{x}) d\vec{x}. \tag{7}$$

为了利用衍射图像确定晶体的密度，需要进行逆傅里叶转换：

$$\rho(\vec{x}) \sim \int F(\vec{q}) \cdot \exp(-i\vec{q}\vec{x}) d\vec{q}. \tag{8}$$

由于我们的反射是离散的，所以上式可以写成以下形式

$$\rho(\vec{x}) \sim \sum |F(\vec{q})| e^{i\varphi} \cdot \exp(-i\vec{q}\vec{x}), \tag{9}$$

其中对所有反射进行求和。事实上，只取最亮的反射就够了，因为它们对总和的贡献最大。

实验测得的强度 $I(\vec{q})$ 只能确定模量 $|F(\vec{q})|$。遗憾的是，强度不包含相位的信息 φ，因此无法从强度直接计算出 $\rho(\vec{x})$。这个问题被称为相位问题。

通常解决相位问题的方法是得到一些近似的相位，然后用这些相位计算晶体的密度，然后用这个密度更新相位，重复这样做，直到得到满意的结果。

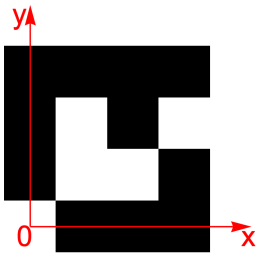
有几种方法可以获得初始相位。一种是假设您已经有了一个已知的晶体结构，并且已知和未知晶体在结构上是相似的。在这种情况下，您可以简单地使用从已知结构中计算出的相位与未知结构的逆傅里叶变换（9）中的测量强度。

... 你将成功解出的

你有 3 个不同的 2D 晶体。MR0, MR1, MR2. 每一个单元格都是一个 4x4 的正方形，有些正方形不透明（$\rho = 0$），有些正方形透明（$\rho = 1$）。原胞 MR0 是已知的（图～ 11）。结构 MR1(可能的变体如图～ 12 所示) 和 MR2(含有 7 个白色方块) 未知，但你知道它们与 MR0 很相似。

使用上述方法，您将确定 MR1 和 MR2 结构。初步计算出 MR0 结构（图 11 所示坐标系）的反射相位（以弧度为单位）$|h|, |k| \leq 2$。

提示：密度的虚部 Im$\rho(\vec{x})$ 可以忽略，这意味着只需考虑 (9) 中每一项的实部即可。



11：MR0 结构与坐标系。白色方块–透明，黑色–不透明。

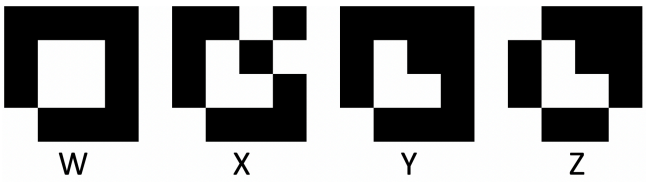


图 12：MR1 晶体的可能原胞

k \ h	-2	-1	0	1	2
2	3.142	0.4636	0	-0.4636	3.142
1	0	-0.4636	2.034	-1.571	0
0	3.142	-1.571	0	1.571	3.142
-1	0	1.571	-2.034	0.4636	0
-2	3.142	0.4636	0	-0.4636	3.142

图 13：MR0 结构的 $|h| \leq 2, \quad |k| \leq 2$ 反射相位（以弧度为单位）。

D.1	晶体（MR0 或 MR2）被强度为 I_0 的光照。求出在 $\vec{q} = 0$ 下的最大强度。	1.0pt
D.2	确定 MR1 晶体的原胞结构。该 MR1 晶体具有图 12 中的一个给定单元格（图）。描述你的答案。	2.0pt
D.3	确定 MR2 晶体的单元格结构。MR2 的结构与 MR0 相似：两个不透明的方块变成了透明的。	2.0pt

设备

设备清单

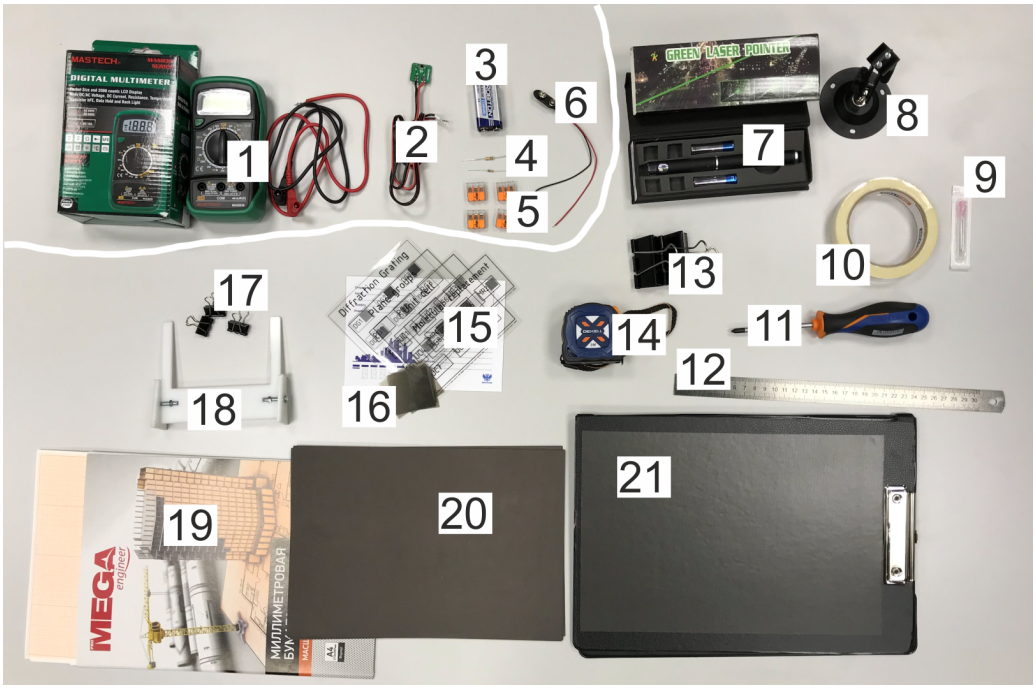


图 14：设备。左上角–强度探测器的细节。

1. 多用表
2. 带磁铁的光电二极管
3. 电池
4. 10 千欧姆和 200 千欧姆的电阻器
5. 拼接连接器 (4 个)
6. 电池夹
7. 红激光（波长 ☒=630 纳米）
8. 激光台
9. 针
10. 胶带
11. 螺丝刀
12. 直尺
13. 大夹子 (2 个)
14. 卷尺
15. 幻灯片（” 衍射光栅”、” 原胞”、” 平面组”、” 分子置换”）与样品
16. 用于降低激光强度的薄膜
17. 小夹子 (4 个)

18. 滑座（需要组装）
19. 绘图纸
20. 卡片
21. 带磁性层的平板

样品

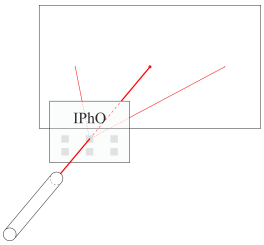


图 15：如图所示放置并照亮样品。

样品应按以下方法用激光照亮。首先定位样品，使标题（幻灯片的英文标题）在幻灯片的顶部，从左到右阅读。在这种情况下，激光应该在幻灯片的前面，而观察的屏幕应该在幻灯片的后面（图 15）。

强度探测器

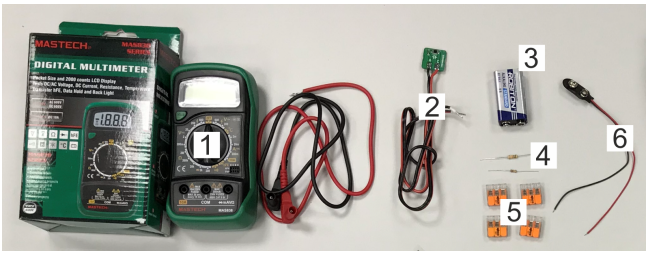


图 16：强度探测器的细节

要测量强度，可根据图 17 所示的电路，用光电二极管与电压表、电阻和电池连接。在这种电路中，通过二极管的电流与入射光的强度成正比。

- 注意连接的极性。在电池块上，红线为正极。默认情况下，使用 10kΩ 电阻，如果测量强度非常小–使用 200kΩ 电阻。

要使用连接器（图 18），请按以下步骤操作。抬起压板，将导线插入连接器，放下压板。三个连接点都是相互电连接的。

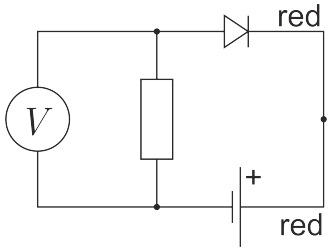


图 17：强度探测器电路。注意连接的极性。



图 18：拼接连接器的方案。所有三个连接器点都是相互电连接的。

激光

将激光器安装在支架上 8. 用一个大夹子把激光器固定上去。如果你需要使激光束变窄，那就用遮蔽胶带封住激光孔，并用针在上面开个洞。图 19 是供你参考的有关内容。

! 注意! 直接暴露在激光束下对眼睛有害。请勿将激光束指向自己或其他人。如果您不使用激光器，请关闭激光器。

! 注意! 锋利的针头很危险：你可能会受伤! 请小心使用。不使用时，请将针头放入保护套中。



图 19：激光器使用大夹子保持开机状态。为了减小光束大小，使用粘性胶带和针。在支架（大夹子）上贴上增黑膜，以降低激光强度。

A. 从狭缝到晶体

A.1 (0.3 pt)

$I(q) =$

A.2 (0.2 pt)

$q =$

A.3 (0.2 pt)

$q(q_1) =$

q_1 和 a 的表达式:

A.4 (1.0 pt)
解释时用到的公式，设计，作图:

DG1: $q_1 =$ $a =$

DG2: $q_1 =$ $a =$

DG3: $q_1 =$ $a =$

DG4: $q_1 =$ $a =$

DG5: $q_1 =$ $a =$

A.5 (1.5 pt)
解释时用到的公式，设计，作图::

DG3: $a/b =$

DG4: $a/b =$

DG5: $a/b =$

A.6 (0.7 pt)

$\rho(x) =$

$F_A(h) =$

$h =$

A.7 (0.7 pt)

$F_B(h) =$

$h =$

A.8 (0.4 pt)

$\frac{I_{A,h=0}}{I_{B,h=0}} =$

$\frac{I_{A,h=1}}{I_{B,h=1}} =$

B. 二维晶体

B.1 (1 pt)

$q_1 =$

$q_2 =$

$\beta =$

B.2 (1.0 pt)

晶体 A $|F(h,k)| =$

晶体 D $|F(h,k)| =$

B.3 (0.6 pt)

$a_{UC1} =$

$a_{UC2} =$

$a_{UC3} =$

$a_{UC4} =$

B.4 (0.4 pt)

UC1 – UC2 – UC3 – UC4 –

B.5 (0.8 pt)

$b =$

B.6 (1.2 pt)

UC5

$a_1 =$

$a_2 =$

$\alpha =$

UC6

$a_1 =$

$a_2 =$

$\alpha =$

UC7

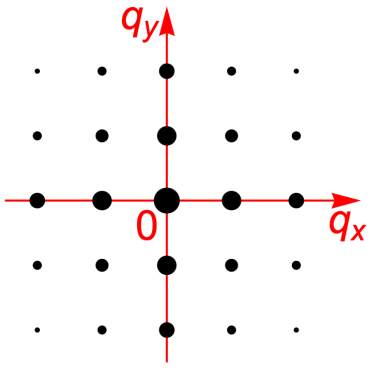
$a_1 =$

$a_2 =$

$\alpha =$

C. 晶体的对称性

C.1 (0.3 pt)



$h =$ $k =$

$m =$

C.2 (0.2 pt)

所有可能的对称轴的方程 (Equations for all possible axis symmetry):

C.3 (0.4 pt)
对称性标示 (C_m 表示转动，以及镜像对称轴的方程) 和光强方程 $I(q_x, q_y)$

C $I(q_x, q_y) =$

C $I(q_x, q_y) =$

C $I(q_x, q_y) =$

$I(q_x, q_y) =$

$I(q_x, q_y) =$

$I(q_x, q_y) =$

$I(q_x, q_y) =$

C.4 (0.2 pt)

$I(-h, -k) =$

对应哪个对称性?

C.5 (0.4 pt)

$f_2(q_x, q_y) =$

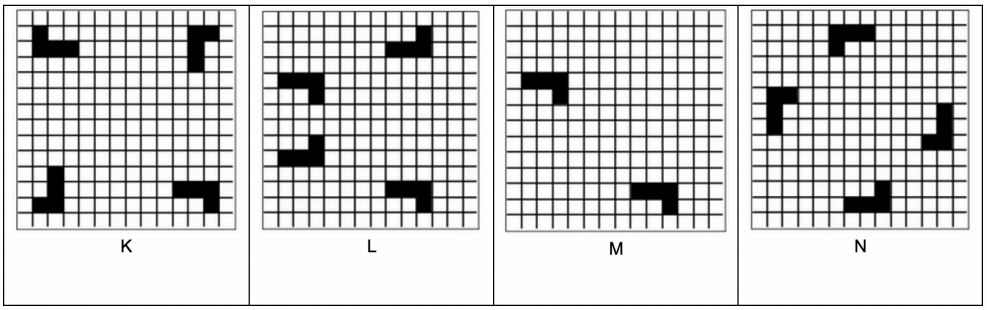
$f_3(q_x, q_y) =$

$f_4(q_x, q_y) =$

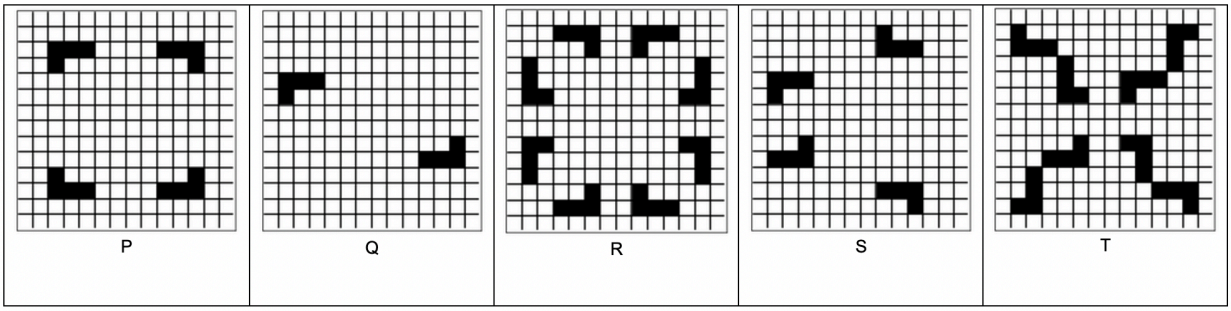
C.6 (0.5 pt)
列出所有可能的转动对称性:

$m =$

C.7 (0.9 pt)



C.7 (cont.)



C.8 (0.8 pt)
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C.9 (1.0 pt)
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C.10 (0.3 pt)

它可能是晶体吗？在选项上画圈: Yes / No

D. 你还需要相位信息...

D.1 (1 pt)

$I_{MR0} =$

$I_{MR2} =$

D.2 (2.0 pt)

MR1:

D.3 (2.0 pt)
MR2:

W1-2

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Experiment

CHN-S-3 E-1 W-3

W1-3

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W1-7

W1-8

W1-9

W1-11

Experiment

CHN-S-3 E-1 W-12

W1-12

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Experiment

CHN-S-3 E-1 W-13

W1-13

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Experiment

CHN-S-3 E-1 W-14

W1-14

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Experiment

CHN-S-3 E-1 W-15

W1-15

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Cubic oscillations

Events

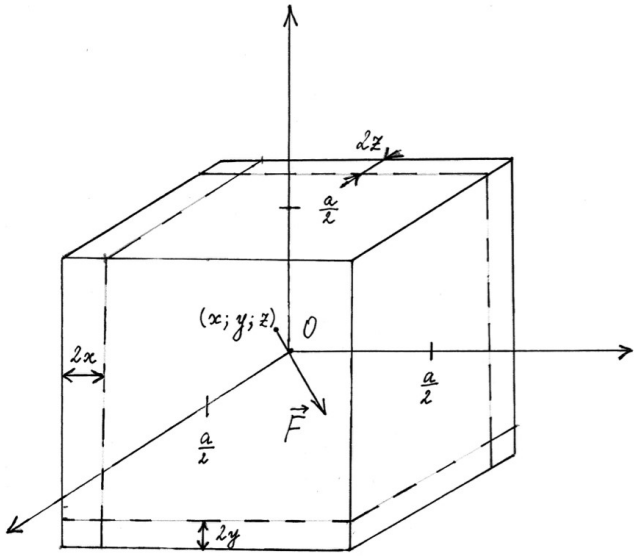
People's Republic of China
Delegation

Решение

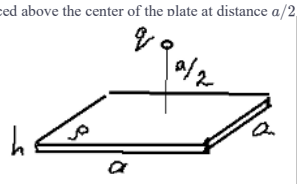
T Условие
S Решение
M Разбалловка

A1 [2.50] A narrow straight channel passes through the center of a fixed cube with a side a . The cube is uniformly charged, the charge density is ρ . The distance from the cube center to the point of intersection of the channel and a face is L . In the channel there is a particle of a mass m and a charge q . Find the period of small oscillations of the particle near the center. The gravitational interaction of the particle and the cube can be neglected. The cube and the particle are oppositely charged.

We will use a coordinate system with axes parallel to the cube's edges, with the origin set at the cube's center. Assuming the particle is at coordinates (x, y, z) $x \ll a$, $y \ll a$, $z \ll a$, we will find the force $\vec{F}$ acting on the particle, by splitting the cube $a \times a \times a$ into a rectangular cuboid $(a-2x) \times (a-2y) \times (a-2z)$ and three square plates of thickness $2x$, $2y$ and $2z$.



The particle is in the center of the cuboid, so there is no force from the cuboid.
Let us find the force between a particle with a charge q and a uniformly charged square plate of small thickness h and edge length a . The plate's charge density is ρ , the particle is placed above the center of the plate at distance $a/2$.



Due to symmetry and Gauss's law, the flux of the particle's electric field through the plate is

$$\Phi = \frac{q}{6\varepsilon_0}.$$

Hence, the force

$$F = \sigma\Phi = \frac{q\rho h}{6\varepsilon_0},$$

where $\sigma = \rho h$ is plate's surface charge density.
Three square plates act on the particle with forces $\vec{F}_1 = \frac{q\rho x}{3\varepsilon_0}\hat{x}$, $\vec{F}_2 = \frac{q\rho y}{3\varepsilon_0}\hat{y}$, and $\vec{F}_3 = \frac{q\rho z}{3\varepsilon_0}\hat{z}$. Net force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \frac{q\rho}{3\varepsilon_0}\vec{r}$, where $\vec{r}$ is the position vector of the particle.

The particle's equation of motion

$$m\ddot{\vec{r}} = \frac{q\rho}{3\varepsilon_0}\vec{r}$$

is an equation of simple harmonic motion with period

$$T = 2\pi\sqrt{\frac{3m\varepsilon_0}{q(-\rho)}}.$$

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Suspension in magnetic field

Решение

T Условие
S Решение
M Разбалловка

B1 [3.00] A current I flows through a loop made of a weightless flexible wire. The loop upper point is attached to the ceiling and a weight is suspended to its lowest point. The half length of the loop is L . The loop is placed in a vertical magnetic field B . The system has reached a stable equilibrium in which the point of suspension at the ceiling and the point of weight suspension are not on the same vertical. Find the wire tension T and the weight P if the distance from the ceiling to the lowest point of the loop is H .

First approach

The Ampere's force acting on a short piece of wire of length $\vec{dl}$ is

$$\vec{F}_{amp} = I\vec{\mathrm{d}l} \times \vec{B}.$$

The net force acting on this piece is

$$\vec{F} = \vec{T}_1 + \vec{T}_2 + I\vec{\mathrm{d}l} \times \vec{B} = 0. \qquad (1)$$

Here $\vec{T}_1$ and $\vec{T}_2$ are tension force of the wire acting on the two end points of the piece.
Projected on the $\vec{\mathrm{d}l}$, this equation reads:

$$T_1 = T_2.$$

Therefore, the tension is constant along the wire.
Let $\vec{n}(l)$ be the tangent vector of the wire at the distance l from the point of suspension. The equation (1) reads

$$\vec{T}_1 + \vec{T}_2 + I\vec{\mathrm{d}l} \times \vec{B} = \mathrm{d}l(T\,\mathrm{d}\vec{n}(l)l + I\vec{n}(l) \times \vec{B}) = 0,$$

$$\mathrm{d}\vec{n}(l)l = -\frac{I}{T}\vec{n}(l) \times \vec{B}.$$

This equation implies that

$$\vec{B} \cdot \mathrm{d}\vec{n}(l)l = 0 \implies \vec{B} \cdot n(l) = \mathrm{const}.$$

The tangent vector of the wire $\vec{n}$ is at constant angle to the magnetic field. In the horizontal plane the tangent vector is rotating at a constant speed. Thus, each side of the loop has the cylindrical helix shape, i. e. it is winding around some cylinder and making the constant angle
\marginnote{\textbf{0.3 pt} - helix shape} α to the magnetic field. To find the radius of this cylinder, let's write the equation in the projection on the horizontal plane:

$$\mathrm{d}\vec{n}_{xy}l_{xy}\sin(\alpha) = -\frac{I}{T}\vec{n}_{xy} \times \vec{B}.$$

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\marginnote{\textbf{0.3 pt} - helix shape} α to the magnetic field. To find the radius of this cylinder, let's write the equation in the projection on the horizontal plane:

$$\mathrm{d}\vec{n}_{xy}l_{xy}\sin(\alpha) = -\frac{I}{T}\vec{n}_{xy} \times \vec{B}.$$

$$R = \frac{T}{IB}\sin(\alpha). \qquad (2)$$

At the point of the weight suspension we have the balance between three forces: two forces of tension(from each side of the loop) and the gravitational force of the weight, which is vertical. These three forces should be in the same vertical plane. Therefore, both sides of the loop are winding around the same cylinder. Each half of the loop should make a half-turn around this cylinder.
Thus, the length of the half of the loop is

$$L = \sqrt{H^2 + (\pi R)^2}.$$

Therefore,

$$R = \frac{1}{\pi}\sqrt{L^2 - H^2} = \frac{1}{\pi}L\sin(\alpha). \qquad (3)$$

The tension force of the wire can be found from equations (2) and (3)

$$T = \frac{IBR}{\sin(\alpha)} = \frac{IBL}{\pi}.$$

The suspended weight is

$$P = 2T\cos(\alpha) = 2T\frac{H}{L}.$$

$$P = \frac{2IBH}{\pi}.$$

Second approach
Stable equilibrium corresponds to a minimum of potential energy. The potential energy is a sum of the gravitational energy of the weight and of the energy of the loop in the magnetic field. Let S be an area of the loop projected on the horizontal plane.

$$E_{\text{p}} = -PH - ISB.$$

E_{p} depends on two variable parameters: H and S . Note that we can change the form of the projection (not changing its length) of the loop onto horizontal plane without changing H . Thus, the minimum of the potential energy corresponds to the maximum area of the projection with fixed length — a circle.
Now, if the projection is fixed, the height H is maximal if the wire makes a constant angle with the vertical axis. Thus, the wire has a shape of cylindrical helix — each side is winding a cylinder, while making constant angle α with vertical axis.
Let R be the radius of the cylinder.
Then

$$S = \pi R^2 = \frac{1}{\pi}(L^2 - H^2).$$

$$E_{\text{p}} = -PH - IB\frac{1}{\pi}(L^2 - H^2).$$

The condition for the minimum of the potential energy is

$$\frac{\partial E}{\partial H} = 0.$$

$$P = \frac{2IBH}{\pi}.$$

The force balance equation at the point of weight suspension:

$$2\frac{H}{L}T = P.$$

$$T = \frac{LIB}{\pi}.$$

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Rod in magnetic field

Решение

T Условие
S Решение
M Разбалловка

C1 [4.50] A weightless rod of a length $2R$ is placed perpendicular to a uniform magnetic field $\vec{B}$. Two identical small balls of mass m and charge q each are attached at the rod ends. Let us direct z-axis along the magnetic field and place the origin at the rod center. The balls are given the same initial velocity v but in opposite directions so that one of the velocities is precisely in the z-direction. What are the maximum coordinates $z_{\max}$ of the balls? Express your answer in terms of q, B, m, v, and R. Find the magnitude of the ball accelerations at this moment and express your answer in terms of q, B, m, v, R, and $z_{\max}$.

—

The equations of motion of the centre of mass of balls is:

$$2m\dot{\vec{v}}_C = q\left[(\vec{v}_1 + \vec{v}_2) \times \vec{B}\right] = q\left[\vec{v}_C \times \vec{B}\right],$$

where $\vec{v}_C$ is the velocity of the centre of mass, $\vec{v}_1$ and $\vec{v}_2$ are velocities of balls.
Notice that $v_C = 0$ at the start of motion then v_C remains equal to 0 always.
The equation for the rotation of the ball about the centre of mass is:

$$\frac{d\vec{L}}{dt} = q\left[\vec{R} \times \left[\vec{v} \times \vec{B}\right]\right].$$

Now let's use the formula of a triple vector product:

$$\left[\vec{R} \times \left[\vec{v} \times \vec{B}\right]\right] = \vec{v}\left(\vec{R} \cdot \vec{B}\right) - \vec{B}\left(\vec{R} \cdot \vec{v}\right).$$

So balls move around the sphere with the center in the middle of rod. It means that $\vec{R} \perp \vec{v}$ then $\vec{R} \cdot \vec{v} = 0$. Hence

$$\frac{d\vec{L}}{dt} = q\vec{v}\left(\vec{R} \cdot \vec{B}\right)$$

Firstly, $\vec{B}$ directs along z-axis, then $\left(\vec{R} \cdot \vec{B}\right) = Bz$, where z is a coordinat of the ball. Secondly, let look at projection of vector equation onto z-axis:

$$\frac{dL_z}{dt} = qv_z Bz = qBz\frac{dz}{dt}.$$

Multiplying to dt and integrating gives us this equation:

$$L_z = \frac{qBz^2}{2} + C,$$

where $C = 0$ because $L_z = 0$ at the start of the motion when $z = 0$.
When z is maximum $v_z = \dot{z} = 0$. It means, that $L = L_z = mur$, where r is a distance between the ball and z-axis, u is the velocity of the ball at this moment.
Recall that the ball is moving around the sphere then $r^2 + z^2 = R^2$.

The Lorenz force and the contact ball-rod force don't do the work. It means the kinetic energy of the system is constant. Together with $\vec{v}_C = 0$, it gives that velocities of the balls don't change i.e. $u = v$.

To summarize, $L_z = mv\sqrt{R^2 - z_{\max}^2}$.
Let's combine the results of the two last blocks as a biquadratic equation for $z_{\max}$:

$$\left(\frac{qB}{2mv}\right)^2 z_{\max}^4 + z_{\max}^2 - R^2 = 0.$$

Solution is:

$$z_{\max} = \frac{\sqrt{2}mv}{qB}\sqrt{\sqrt{1 + \left(\frac{qBR}{mv}\right)^2} - 1}$$

This result can be obtained in a faster way by those who are familiar with the generalized momentum of a charged particle in a magnetic field,

$$\vec{\mathcal{P}} = m\vec{v} + q\vec{A},$$

where $\vec{A}$ is the vector potential which is defined by the condition $\mathrm{curl}\vec{A} = \vec{B}$. We can choose $\vec{A} = \frac{1}{2}\hat{\tau}Br$, where r denotes the distance from the z -axis, and $\hat{\tau}$ stands for the unit vector perpendicular both to the radius vector and to the z -axis. This is a vector field obeying cylindrical symmetry around the z -axis and because of that, the corresponding generalized momentum is conserved,

$$\mathcal{L}_z = \vec{R} \times \vec{\mathcal{P}} \cdot \hat{z} = mrv_\perp + \frac{1}{2}qBr^2 \equiv \frac{1}{2}qBR^2;$$

here $v_\perp$ denotes the velocity component perpendicular to the z -axis. Noting that at the topmost position of the ball's trajectory, $v_\perp = v$ due to the fact that the speed is constant and horizontal (i.e. in $x-y$ -plane), we obtain a quadratic equation for $\rho = \frac{r}{R}$:

$$\rho^2 + 2\kappa\rho - 1 = 0, \quad \kappa \equiv \frac{mv}{qBR}.$$

We select the positive root $\rho = \sqrt{\kappa^2+1} - \kappa$. Now we can write the final answer as

$$z_{\max} = R\sqrt{1-\rho^2} = R\kappa\sqrt{2}\sqrt{\sqrt{\kappa^{-2}+1}-1}$$

which coincides with the previous result.
Consider three orthogonal unit vectors: 1-st is along with the velocity, 2-nd along with direction from ball to the middle of the rod, and 3-rd is perpendicular to the other two.

As we found earlier that $v = \mathrm{const}$, so $a_1 = 0$. Further, a_2 coinsides with v^2/R because ball is moving around sphere. Finally, the projection of 2-nd Newton law onto the 3-rd axis is: $ma_3 = qvB\frac{z_{\max}}{R}$. Hence

$$a = \sqrt{\left(\frac{v^2}{R}\right)^2 + \left(\frac{qvBz_{\max}}{mR}\right)^2}$$

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Решение

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A1 [0.50] At what angle α_1 to the X axis should the body velocity vector be for the absolute value of the power of the friction force be at maximum?

—

Power of the friction force is given by the equation:

$$P = (F, v) = -\left(\mu_x mg\cos^2\alpha + \mu_y mg\sin^2\alpha\right) v$$

since $\mu_x > \mu_y$, then maximum of the power is reached when $\alpha = 0$

Ответ:

$$\alpha = 0$$

A2 [0.50] At what angle α_2 to the X axis should the body velocity vector be for the absolute value of the power of the friction force be 1.2 times less than maximum?

—

From the previous point it follows that:

$$\mu_x mgv\cos^2\alpha + \mu_y mgv\sin^2\alpha = \frac{1}{1.2}\mu_x mgv$$

From where we get:

$$\cos\alpha_2 = \pm\sqrt{\frac{5\mu_x - 6\mu_y}{6(\mu_x - \mu_y)}} = \pm\frac{1}{\sqrt{2}}$$
$$\sin\alpha_2 = \pm\sqrt{\frac{\mu_x}{6(\mu_x - \mu_y)}} = \pm\frac{1}{\sqrt{2}}$$

Ответ:

$$\alpha_2 = \pm\pi/4; \pm 3\pi/4$$

A3 [1.00] Let the initial velocity have components $v_{0x} = 1\text{m/s}$ and $v_{0y} = 1\text{m/s}$. After some time the velocity component along the Y axis equals $v_{1y} = 0{,}25\text{m/s}$. What is the velocity magnitude at this moment?

—

Newton's equation:

$$\begin{cases} m\dfrac{dv_x}{dt} &= -\mu_x mg\dfrac{v_x}{v} \\ m\dfrac{dv_y}{dt} &= -\mu_y mg\dfrac{v_y}{v} \end{cases}$$

Dividing one equation by another, we get:

$$\frac{dv_x}{dv_y} = \frac{\mu_x v_x}{\mu_y v_y}$$

Integrating these equations, we obtain:

$$\frac{v_x^{\mu_y}}{v_y^{\mu_x}} = Const$$

From the initial conditions we find:

$$v_x = 0.125\text{m/s}$$

So the absolute value of velosity is

Ответ:

$$v = 0.28\text{m/s}$$

A4 [1.00] Let the velocity be $v_2 = 1.0\text{m/s}$. At what angle α_3 to the X axis should the velocity vector be for the radius of curvature of the trajectory be minimum? What is this radius equal to? The free fall acceleration is $g = 9{,}8\text{m/s}^2$.

—

Projection of the friction force onto the direction perpendicular to the velocity is given by equation:

$$F_n = mg(\mu_x - \mu_y)\sin\alpha\cos\alpha$$

Newton's second law projected onto the direction perpendicular to the velocity has the form:

$$\frac{mv^2}{R} = mg(\mu_x - \mu_y)\sin\alpha\cos\alpha,$$

where R is the radius of curvature of the trajectory. It's obvious that

$$R = \frac{v^2}{g(\mu_x - \mu_y)\sin\alpha\cos\alpha} = \frac{2v^2}{g(\mu_x - \mu_y)\sin 2\alpha}$$

The minimum radius of curvature will be when:

Ответ:

$$\alpha = \pi/4$$

and

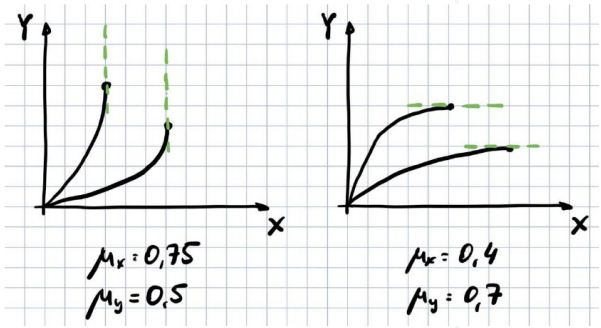
$$R_{min} = \frac{2v^2}{g(\mu_x - \mu_y)}$$

In a single diagram on the XY plane, sketch the trajectories of the body

launched at the angles $\alpha_4 = \pi/6$ and $\alpha_5 = \pi/3$ for the friction coefficients specified above. The magnitudes of initial velocities are the same. Solve the same problem for the friction coefficients $\mu_x = 0{,}4$ and $\mu_y = 0{,}7$.

If $\mu_x = \mu_y$, then the accelerations along the x and y axes will be proportional to the ratio of the initial velocities and the body will move in a straight line. If $\mu_x > \mu_y$, then the speed along the x-axis will decrease faster than in the previous case and the body will deviate from a linear motion as shown in the figure. The direction of deflection does not depend on how the initial velocity is directed. If $\mu_x < \mu_y$, then the body will deviate in the opposite direction.

Ответ:



A body of mass m is at rest at the origin. A force has been applied to it

at an angle α to the X axis. The force magnitude $F(t) = \gamma t$ linearly grows with time. Find the dependence of the moment the body starts moving on α . Ignore the stagnation phenomenon.

—

Let us notice that the body will start the motion with the angle $\varphi \neq \alpha$. Then projections of the forces at this moment are given by

$$\begin{cases} F_{\text{fr},x} = \mu_x mg\cos\varphi = \gamma t\cos\alpha \\ F_{\text{fr},y} = \mu_y mg\sin\varphi = \gamma t\sin\alpha \end{cases}$$

Dividing these equations by $\mu_x mg$ and $\mu_y mg$ correspondingly and summing up squares, we get

$$(mg)^2 = (\gamma t)^2\left[\left(\frac{\cos\alpha}{\mu_x}\right)^2 + \left(\frac{\sin\alpha}{\mu_y}\right)^2\right]$$

So the answer is

Ответ:

$$t = \frac{\mu_x\mu_y mg}{\gamma\sqrt{\mu_y^2\cos^2\alpha + \mu_x^2\sin^2\alpha}}$$

For a given initial velocity v_0 find the dependence of its velocity v on the

angle of rotation of the rod φ assuming that the other body remains at rest.

—

Newton's law projected onto the direction of motion of a point-like mass is:

$$m\frac{dv}{dt} = -mg(\mu_x \cos^2\varphi + \mu_y \sin^2\varphi)$$

Since $v = L\dot{\varphi}$, then:

$$L\ddot{\varphi} = -g(\mu_x \cos^2\varphi + \mu_y \sin^2\varphi)$$

Multiply this equation by $\dot{\varphi}\,dt$ we get

$$L\dot{\varphi}d\dot{\varphi} = -g(\mu_x \cos^2\varphi + \mu_y \sin^2\varphi)d\varphi$$

Integrating this equation, we obtain:

Ответ:

$$v^2 + gL\left((\mu_x + \mu_y)\varphi + (\mu_x - \mu_y)\frac{\sin 2\varphi}{2}\right) = v_0^2$$

Find the maximum value of the initial velocity $v_{0\max}$ at which the other

body will remain at rest.

—

Newton's law for a moving body in projection onto a rod has the form:

$$T + F_{\text{fr}} \sin\beta = m\dot{\varphi}^2 L$$

where β is the angle between frictional force and direction of velocity.
Using the result from C1, we find that:

$$T = \frac{mv_0^2}{L} - mg(\mu_x + \mu_y)\varphi - mg(\mu_x - \mu_y)\sin 2\varphi$$

Using the result from B1, we find that:

$$\frac{mv_0^2}{L} - mg(\mu_x + \mu_y)\varphi - mg(\mu_x - \mu_y)\sin 2\varphi \le \frac{\mu_x\mu_y}{\sqrt{\mu_y^2\sin^2\varphi + \mu_x^2\cos^2\varphi}}mg$$

Note that with increasing φ , the left hand side decreases, and the right hand side increases, so the body starts to move right at $\varphi = 0$. So

Ответ:

$$v_{0\max} = \sqrt{\mu_y g L} = 2.2\text{m/s}$$

What distance will the body travel until it stops completely if the initial

velocity is $v_{0\max}$?

—

Substituting $v_{0\max}$ from the previous section in the final equation of the C1, we get the equation for the φ at the stopping point $(v = 0)$:

$$gL\left((\mu_x + \mu_y)\varphi + (\mu_x - \mu_y)\frac{\sin 2\varphi}{2}\right) = \mu_y g L$$

Numerically solving this equation, we get the answer for the φ , and so the travelled distance is equal to

Ответ:

$$L\varphi = 0.34\text{m}$$

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Решение

T Условие

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A1 [1.00] An atom emits light with a wavelength $\lambda_0 = 300\text{nm}$. Using the classical model estimate an emission time τ (that is, the period of time it takes the atom to emit the energy equal to that of a single photon). This time coincides with the characteristic time, during which the atom emits a photon, by the order of magnitude. All radiation is due to a single electron located at a distance about $a_0 = 0.1\text{nm}$ from the nucleus. Express your answer in terms of the physical constants, λ_0 , and a_0 .

—

The power of radiation is

$$W = \frac{2k}{3c^3}\langle \ddot{\vec{p}}(t)^2\rangle = \frac{2k}{3c^3}\langle \omega^4 p_m^2 \cos^2(\omega t + \varphi)\rangle = \frac{k\omega^4 p_m^2}{3c^3}.$$

Here p_m is the dipole moment amplitude which can be estimated as $p_m \simeq ea_0$. Time of radiation is

$$\tau = \frac{\hbar\omega}{W} \approx \frac{3\hbar}{ka_0^2 e^2}\left(\frac{\lambda_0}{2\pi}\right)^3 \approx 1.5 \cdot 10^{-8} s.$$

Here we used the relation $\omega = 2\pi c/\lambda_0$.

Ответ:

$$\tau \approx \frac{3\hbar}{ka_0^2 e^2}\left(\frac{\lambda_0}{2\pi}\right)^3 \approx 1.5 \cdot 10^{-8} s.$$

A2 [0.25] Estimate the power W_s of electromagnetic radiation of all N atoms in the spontaneous emission mode, i.e. when the direction of atomic dipole and the phase of its oscillations randomly change from atom to atom. In your answer write down the formula for the power in terms of N, ω , and τ .

—

System of the atoms' dipole moment is

$$\vec{P}(t) = \sum_i \vec{p}_{mi} \cos(\omega t + \varphi_i).$$

The power of radiation is

$$W = \frac{2k}{3c^3}\langle \ddot{\vec{P}}(t)\rangle = \frac{k\omega^4}{3c^3}\sum_i p_{mi}^2 + \frac{k\omega^4}{3c^3}\sum_{i\neq j}(\vec{p}_{mi}\cdot\vec{p}_{mj})\cos(\varphi_i - \varphi_j).$$

We used $\langle \cos^2(\omega t + \varphi)^2\rangle = \frac{1}{2}$ and $\langle \cos(\omega t + \varphi_1)\cos(\omega t + \varphi_2)\rangle = \frac{1}{2}\cos(\varphi_i - \varphi_j)$. All dipole moments have the same magnitude, are oriented randomly and their phases are not correlated. Therefore the second sum equals zero. The first sum consists of N equal terms and te power is

$$W_s = N\frac{k\omega^4}{3c^3}p_m^2.$$

For one atom we have

$$\frac{k\omega^4 p_m^2}{3c^3} = \frac{\hbar\omega}{\tau}$$

and thus

Ответ:

$$W_s = N\frac{\hbar\omega}{\tau}.$$

A3 [0.25] Estimate the duration of the spontaneous emission pulse of this system of atoms. Express your answer in terms of the same quantities.

—

The energy emitted by all atoms is $E = N\hbar\omega$. Hence the time of radiation is

$$\Delta t_s = \frac{E}{W_s} \approx \tau.$$

Ответ: $\Delta t_s = \tau$.

A4 $^{0.50}$ Estimate the power W_i of electromagnetic radiation of all N atoms in the superradiance mode, i.e. when the direction of atomic dipoles and the phases of their oscillations are the same for all atoms in the excited state. Express your answer in terms of N, ω , and τ .

—

In this case all dipole moments are oriented in the same direction and oscillate with the same phase $\varphi_i = \varphi_j$. All terms in both sums in the expression for power in A2 are the same. The first sum has N terms and the second has $N(N-1)$ terms. The power is

$$W_i = (N + N(N-1))\frac{k\omega^4}{3c^3}p_m^2 = N^2\frac{\hbar\omega}{\tau}.$$

Ответ:

$$W_i = N^2\frac{\hbar\omega}{\tau}.$$

Estimate the duration of the radiation pulse of the system of atoms in the

superradiance mode. Express your answer in terms of the same quantities.

—

Emitted energy is the same as in A3. The time of radiation

$$\Delta t_i = \frac{E}{W_i} \approx \frac{\tau}{N}.$$

Ответ: $\Delta t_i = \frac{\tau}{N}$.

Let the amplitudes of two wave maxima be E_{m1} and E_{m2}. Find the

difference in their propagation speeds Δv . Express your answer in terms of n_0, n_2, c, E_{m1} , and E_{m2} .

—

Wave propagation speed is

$$v = \frac{c}{n} = \frac{c}{n_0 + n_2 E_m^2} \approx \frac{c}{n_0}\left(1 - \frac{n_2 E_m^2}{n_0}\right).$$

So the speed difference is

Ответ:

$$\Delta v = v_1 - v_2 = \frac{cn_2}{n_0^2}\left(E_{m2}^2 - E_{m1}^2\right).$$

A light pulse with a wavelength in vacuum of λ_0 = 300nm and a

maximum intensity of $I_0 = 3 \cdot 10^9 \mathrm{W/cm^2}$ propagates along the axis of a quartz fiber. Assume the envelope of a time dependence of the electric field squared $E_m^2(t)$ of the wave to be a parabola. How far (find the distance s) does the pulse propagate along the fiber before its spectral width increases by the factor of $K = 200$? Express your answer in terms of K, λ_0, n_2, E_m and calculate the numerical value (in meters, rounded to an integer).

First approach.
The intensity is proportional to the square of electric field amplitude. Pulse envelope equation is

$$E_m^2(t) = E_0^2\left(1 - \frac{4(t-t_0)^2}{(\Delta t_0)^2}\right).$$

Here t_0 is a time when the center of the wavepocket crosses given point in space. Value of t_0 depends on the coordinate along the wave. Electric field in the wave is equal to $E_x(t, z) = E_m(t)\cos\varphi(t, z)$, where the coordinate z is measured along the direction of wave propagation, and the phase is

$$\varphi(t, z) = \omega_0 t - n_0\frac{\omega_0}{c}z - n_2 E_m^2(t)\frac{\omega_0}{c}z.$$

We neglect initial frequency difference because spectral width increases by hundreds of times. In this way, nonlinear effects lead to additional phase shift, proportional to the square of $(t - t_0)$:

$$\delta\varphi = n_2 E_0^2\left(1 - \frac{4(t-t_0)^2}{(\Delta t_0)^2}\right)\frac{\omega_0}{c}z.$$

Angular frequency is the the rate of phase change with time. So quadratic phase shift leads to linear frequency shift

$$\delta\omega(t, z) = \frac{\partial\delta\varphi}{\partial t} = -8n_2 E_0^2\frac{\omega_0}{c}z\frac{t-t_0}{(\Delta t_0)^2}.$$

Spectral width of the pulse after propagating the distance z in the medium is equal to the difference of frequencies at $t = t_0 \pm \Delta t_0/2$:

$$\Delta\omega = -\frac{8}{\Delta t_0}n_2 E_0^2\frac{\omega_0}{c}z\Delta\omega_0.$$

To obtain this formula we used $\Delta\omega_0 \approx 2\pi/\Delta t_0$. The ratio of spectral width at distance $z = s$ and initial spectal width is

$$K = \frac{|\Delta\omega|}{\Delta\omega_0} = \frac{4}{\pi}n_2 E_0^2\frac{\omega_0}{c}s,$$

Therefore the required distance is

$$s = \frac{\pi K c}{4n_2 E_m^2\omega_0} = \frac{K\lambda_0}{8n_2 E_m^2}.$$

To calculate the numerical value of s , one should remember that the peak pulse intensity value is three times higher than the one mentioned in the problem description. Thus the value of $n_2 E_m^2$ is three times higher: $n_2 E_m^2 \simeq 9.6 \cdot 10^{-7}$.
Second approach.
Let us consider the propagation of two adjacent electric field maxima. Due to nonlinearity this maxima move with slightly different speed:

$$\delta v \approx \frac{cn_2}{n_0^2}(E_{m2}^2 - E_{m1}^2).$$

For the given pulse duration and mean frequency it consists of $N = \Delta t_0/T_0 = \omega_0/\Delta\omega_0$ oscillations. The electric field maxima can be enumerated from $-N/2$ to $N/2$. Then the amplitude of electric field maximum with number l is (the maximum with large number arrives at a given point later):

$$E_m^2(l) = E_0^2\left(1 - \frac{4l^2}{N^2}\right).$$

Consequently, the speed difference between adjacent maxima with numbers l and $l + 1$ equals

$$\Delta v_l = v_{l+1} - v_l \approx \frac{cn_2}{n_0^2}E_0^2\frac{8l}{N^2}.$$

Initial frequency is equal ω_0 . After propagation over distance s the frequency of the maximum with number l is:

$$\omega_l = \frac{2\pi}{T_l} = \frac{2\pi}{2\pi/\omega_0 + s/v_{l+1} - s/v_l} = \frac{\omega_0}{1 - s\Delta v_f \omega_0/(2\pi v_l^2)} = \omega_0 + \frac{\omega_0^2}{2\pi v_l^2}\frac{cn_2}{n_0^2}E_0^2\frac{8l}{N^2}s.$$

Frequency linearly depends on the maximum number. The maximal frequency difference is between the first and the last maxima and equals

$$\Delta\omega = \frac{\omega_0^2}{2\pi v_l^2}\frac{cn_2}{n_0^2}\frac{8}{N}s.$$

Hence ($v_l \approx v = c/n_0$ and $\omega_0 = N\Delta\omega_0$)

$$K = \frac{\Delta\omega}{\Delta\omega_0} = \frac{8n_2E_m^2}{\lambda_0}s,$$

and the answer is the same.

Ответ:

$$s = \frac{K\lambda_0}{8n_2E_m^2} \approx 7.8m.$$

What sign should the constant β_2 have in order for the pulse chirped

according to the scheme described above to be compressed in time in this medium? Please, indicate "+" or "-" in your answer. In what follows consider that β_2 has exactly this sign.

To squeeze the chirping pulse, the relatively high-frequency tail of the pulse should propagate faster than its low-frequency 'head'. This is called 'anomalous dispersion'. Speed of the wave propagation is the group velocity

$$V_g = \frac{d\omega}{dk} = \frac{1}{\beta_1 + \beta_2(\omega - \omega_0)},$$

Anomalous dispersion (larger group speed at larger frequencies) is observed for $\beta_2 < 0$.

Ответ: $\beta_2 < 0$.

A pulse described in B2 has a duration $\Delta t_0 = 10$ps and an initial spectral

width $\Delta\omega_0 \approx 2\pi/\Delta t_0$ (before chirping) and propagates in the medium described above. Find the distance the pulse should travel in order to achieve the minimum possible duration after chirping with spectrum broadening by the factor of $K = 200$. Express your answer in terms of physical constants, K, Δt_0, β_1 , and β_2 and calculate the numerical value in meters, rounded to an integer.

—

Since the term associated with β_2 is much less in absolute value than β_1 for the given range of frequencies, we can use approximate expression for the group velocity

$$V_g \approx \frac{1}{\beta_1} - \frac{\beta_2}{\beta_1^2}(\omega - \omega_0).$$

The length of the chirping pulse that just has entered into the considered medium is $\Delta l = V_g \Delta t_0 = \frac{1}{\beta_1}\Delta t_0$. The difference in group velocity of the 'tail' and the 'head' is

$$\Delta V_g \approx \frac{|\beta_2|}{\beta_1^2}\Delta\omega = \frac{|\beta_2|}{\beta_1^2} K \frac{2\pi}{\Delta t_0}.$$

The time in which the 'tail' will catch up with the 'head' up (which corresponds to the minimal duration of the pulse) is

$$t = \frac{\Delta l}{\Delta V_g} = \frac{\beta_1}{2\pi K|\beta_2|}(\Delta t_0)^2.$$

Since $V_g \approx \frac{1}{\beta_1}$ in this period of time the pulse will pass the distance

$$l = V_g t \approx \frac{1}{2\pi K|\beta_2|}(\Delta t_0)^2 \approx 4m.$$

Ответ:

$$l = \frac{1}{2\pi K|\beta_2|}(\Delta t_0)^2 \approx 4m.$$

Nonlinearity of a medium leads to disappearance of diffraction of a light

beam of sufficiently high intensity. Estimate the minimum power of a light pulse W_c at which it does not experience diffraction, i.e. propagates inside a narrow cylindrical channel of constant radius. Express your answer for W_c in terms of physical constants, frequency ω_0 , n_0 , and n_2 . Assume the intensity distribution over the channel cross section to be approximately uniform. Find the numerical value of the power for a pulse with a wavelength in vacuum λ_0 = 300nm propagating in quartz. Coefficient n_0 = 1.47 .

Let be *a* radius of the channel. Diffraction divergence angle of the beam can be estimated as

$$\theta_d \approx \frac{\lambda}{\pi a} = \frac{2c}{n_0 \omega_0 a}.$$

However, due to nonlinearity, the refractive index inside the channel $n = n_0 + n_2 E_m^2$ is larger than refractive index $n = n_0$ outside of it. The beam deflecting from the channel's axis due to diffraction will not get out of it if its angle of incidence will be greater than the angle of total internal reflection α_c . So the critical power corresponds to the condition $\theta_d = \frac{\pi}{2} - \alpha_c \equiv \theta_c$.
Since

$$\cos\theta_c \approx 1 - \frac{\theta_c^2}{2} = \sin\alpha_c = \frac{n_0}{n_0 + n_2 E_m^2} \approx 1 - \frac{n_2 E_m^2}{n_0},$$

we have the following condition

$$E_m^2 = \frac{n_0}{2n_2}\theta_d^2 = \frac{2c^2}{n_0 n_2 \omega_0^2}\frac{1}{a^2}.$$

Corresponding value of the intensity is

$$I_c = \frac{\varepsilon_0 n_0 c}{2} E_m^2 = \frac{\varepsilon_0 c^3}{n_2 \omega_0^2 a^2}.$$

Finally, 'critical' power is

$$W_c = I\pi a^2 = \frac{\pi \varepsilon_0 c^3}{n_2 \omega_0^2}$$

and does not depend on the channel diameter.
The numerical value of the power for the pulse of wavelength λ_0 in a vacuum is equal to

$$W_c = \frac{2\pi n_0 \lambda_0^2 I_1}{3.2 \cdot 10^{-7}} \approx 26MW.$$

Comment: The expression for the diffraction divergence of the circular beam is approximate. Therefore, it is possible to use alternative estimations, for instance $\theta_d \approx \lambda / a$, or $\theta_d \approx \lambda / 2\pi a$.

Ответ:

$$W_c = \frac{\pi \varepsilon_0 c^3}{n_2 \omega_0^2} = \frac{2\pi n_0 \lambda_0^2 I_1}{3.2 \cdot 10^{-7}} \approx 26MW.$$

Propose a method that would allow one to detect an exoplanet with a

noticeable inclination of its orbital plane with respect to the line of sight by means of studying the spectrum of its star in the optical range. As an answer name the physical phenomenon underlying your method.

—

The most natural way is to detect the Doppler shift of some element (e.g. hydrogen) spectral line. The shift is caused by the star rotation with respect to the star-exoplanet system barycenter. This shift should be observed with the exoplanet rotation period.

Ответ: Doppler effect

Suppose an exoplanet of mass m orbits a star of mass M in a circular

orbit of a radius R and the period of revolution is T. The orbital plane is at an angle θ to the direction to Earth. Estimate the accuracy of the relative frequency measurement, $\Delta\omega/\omega$, required to detect such an exoplanet by your method. In your answer express $\Delta\omega/\omega$ in terms of the fundamental constants, R, T, θ, m, and M.

—

Planet's orbital motion velocity is $v = \frac{2\pi R}{T}$. The corresponding velocity of the star is $v_1 = \frac{vm}{M}$. The Doppler effect is sensitive to the velocity sightline component only (the nonrelativistic case is considered). Its maximum value is $v_1\cos(\theta)$. Thus the relative frequency shift is

Ответ:

$$\frac{\Delta\omega}{\omega} = \frac{2\pi R}{Tc}\frac{m}{M}\cos\theta.$$

Assume the mass of the exoplanet and its star to be equal to the mass of

Earth and the Sun, respectively. Assume the radius of the circular orbit to be equal to the distance from Earth to the Sun ($R \approx 1.5 \cdot 10^{11}$m), the angle $\theta = 60^\circ$. The Solar mass is 330,000 times of the Earth's mass, the period of the Earth's revolution around the Sun is 1 year. Find an integer n such that 10^{-n} is the accuracy of relative frequency measurement required by your method. Usage of ultrashort (femtosecond) laser pulses makes it possible to measure frequencies in the optical range (10^{15}Hz) with an accuracy of about 10 Hz. Is this accuracy enough to register the exoplanet?

Even for $\theta = 60^o$ the star velocity sightline component is at most half v_1 . In order to estimate this component, one could assume the exoplanet orbit is circular. Since the same assumption is valid for the Earth's orbit, the maximal sightline projection is

$$v_m = \frac{2\pi R}{T}\frac{1}{330000}\cos\theta \approx 4.5\frac{sm}{s}.$$

The corresponding frequency shift is of order $\frac{\Delta\omega}{\omega} \approx \frac{v_m}{c} \approx 10^{-10}$. Thus, a frequency variation in the tenth significant figure should be detected once a year. The accuracy of an optical frequency comb is of order $\frac{10Hz}{10^{15}Hz} \approx 10^{-14}$. So, exoplanet detection is quite possible.

This method is indeed applied in practice. However a number of difficulties had to be overcome to actually use this technology. In fact, most of the stably generated femtosecond pulses have carrier frequencies outside the visible spectrum, in the microwave and infrared regions. In order to use the method in visible spectrum, a different nonlinear phenomenon should be used -- higher harmonic generation in nonlinear medium. Moreover, to increase the spectral device resolution, frequency filtering should be used which increases the narrow frequency intervals between the lines.

Ответ: $n = 10$, exoplanet detection is possible.

2020 -- We are what they grow beyond.

Решение

T **Условие**

S **Решение**

M **Разбалловка**

A1 0.30 Write the formula 1 using the scattering vector q ($q \ll k_i$).

Ответ:

$$I = \frac{I_0}{N^2} \cdot \left(\frac{\sin(Nqa/2)}{\sin(qa/2)}\right)^2 \cdot \left(\frac{\sin(qb/2)}{(qb/2)}\right)^2$$

A2 0.20 Find scattering vector q for the maximum numbered h for a diffraction grating with a period a.

Ответ:

$$q = \frac{2\pi}{a} \cdot h, \quad h \in \mathbb{Z}$$

DISCLAIMER. Here and below we use the notations:
$\mathbb{Z}$ – integer numbers: $\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{N}$ – positive numbers: $\{1, 2, \dots\}$

A3 0.20 Let q_1 be the scattering vector for the first maximum. Express q in terms of q_1 for intensity maxima. How are q_1 and a related?

Ответ:

$$q = q_1 \cdot h, \quad h \in \mathbb{Z}$$

$$q_1 \cdot a = 2\pi$$

A4 1.00 Observe the diffraction of samples DG1-DG5. Determine experimentally q_1, and a for each sample. Draw a scheme of your setup, write the quantities you measure, and write down the formulas for the calculations.

$q_1 = \frac{2\pi}{\lambda}\frac{s_N}{NL}$,
where s_N - distance between zero maximum and maximum N; L - distance between DG and screen.

Ответ: DG1: $q_1 = 320 \pm 32\text{mm}^{-1}$ $a = 20 \pm 2\mu\text{m}$
DG2: $q_1 = 130 \pm 13\text{mm}^{-1}$ $a = 50 \pm 5\mu\text{m}$
DG3: $q_1 = 79 \pm 8\text{mm}^{-1}$ $a = 80 \pm 8\mu\text{m}$
DG4: $q_1 = 79 \pm 8\text{mm}^{-1}$ $a = 80 \pm 8\mu\text{m}$
DG5: $q_1 = 79 \pm 8\text{mm}^{-1}$ $a = 80 \pm 8\mu\text{m}$

A5 1.50 Conduct an experiment and determine the a/b ratio for samples DG3, DG4, DG5. Explain your solution using formulas, diagrams, and pictures. It is known that $b < a/2$

that $b \leq a/2$.

From formula A1 one could note that intensity I of maximum depends on its number h:

$$I(h) = I_0 \left(\frac{\sin(\pi b/a \cdot h)}{(\pi b/a \cdot h)} \right)^2.$$

Measured intensities for two different maxima gives us a transcendental equation

$$\pi b/a = \arcsin\left(\frac{1}{2} \sqrt{\frac{I(1)}{I(2)}} \sin(2\pi b/a) \right).$$

Root of this equation ($\pi b/a$) could be found numerically.

For DG3 measured intensities are $I(1) = 790mV, \quad I(2) = 630mV$, and solution of equation is $\pi b/a = 0.397$. This leads to $a/b \approx 8.3$, which is quite close to theoretical value 8.

For DG4 and DG5 one could note that each 4th and each 2nd maxima (respectively) are dimmed. That's why $a/b = 4$ for DG4, and $a/b = 2$.

Ответ: DG1: $a/b = 2$
DG2: $a/b = 4$
DG3: $a/b \in [7; 10]$
DG4: $a/b \in [3.2; 4.8]$
DG5: $a/b \in [1.5; 2.5]$

A6 0.70 Write down $\rho(x)$ for the unit cell of the diffraction grating from A1 (Fig. 4A). Use the coordinate system as shown in the figure. Suppose that the unit cell is such that the period of the lattice a is p times the width of the slit b: $a = pb, \quad p \in \mathbb{N}$. Calculate the structure factor $F_A(h)$ for this unit cell for reflex h. Record your answer using h and q_1. What maxima have intensity 0? Write the equation for h for such maxima.

Ответ:

$$\rho(x) = \begin{cases} 1, & x \in [0, b) \\ 0, & x \in [b, a) \end{cases}$$

Ответ:

$$F_A(h) = 2 \cdot \frac{\sin(\pi h/p)}{q_1 h} \cdot e^{i\pi h/p} = \begin{cases} \frac{2\pi}{q1} \cdot \left(\frac{1}{p} \right), & h = 0 \\ F_1(h), & h \neq 0 \end{cases}$$

Ответ: Equation for zero reflections:

$$h = \pm pm, \quad m \in \mathbb{N}$$

A7 0.70 Consider another unit cell (Fig. 4B) of the diffraction grating. Calculate the structure factor $F_B(h)$ for this unit cell for reflex h. What reflexes of this diffraction grating have an intensity of 0? Write the equation for h for such reflexes.

Ответ:

$$F_B(h) = 2 \cdot \frac{\sin(\pi h)}{q_1 h} \cdot e^{i\pi h} - 2 \cdot \frac{\sin(\pi h/p)}{q_1 h} \cdot e^{i\pi h/p} = \begin{cases} \frac{2\pi}{q1} \cdot \left(1 - \frac{1}{p} \right), & h = 0 \\ -F_1(h), & h \neq 0 \end{cases}$$

Ответ: Equation for zero reflections:

$$h = \pm pm, \quad m \in \mathbb{N}$$

A8 0.40 These two diffraction gratings described above are illuminated with light of the same intensity. Find the quotients $I_{A,h=0}/I_{B,h=0}$ and $I_{A,h=1}/I_{B,h=1}$.

Ответ:

$$\frac{I_{A,h=0}}{I_{B,h=0}} = \left(\frac{b}{a-b}\right)^2 = \left(\frac{1}{p-1}\right)^2$$

Ответ:

$$\frac{I_{A,h=1}}{I_{B,h=1}} = 1$$

B1 1.00 Find the angle β between the vectors $\vec{q}_1$ and $\vec{q}_2$ and their lengths q_1, q_2. Please note that these vectors must be of minimum length and the angle between them must be $\leq 90^\circ$. Express your answer through the crystal parameters a_1, a_2, α (Fig. 5)

Ответ:

$$|\vec{q}_1| = \frac{2\pi}{a_2 \sin\alpha}, \quad |\vec{q}_2| = \frac{2\pi}{a_1 \sin\alpha}$$

or

$$|\vec{q}_1| = \frac{2\pi}{a_1 \sin\alpha}, \quad |\vec{q}_2| = \frac{2\pi}{a_2 \sin\alpha}$$

Ответ:

$$\beta = \alpha$$

B2 1.00 For crystals A and D, find the complex amplitude modulus $|F(h,k)|$ for the reflex (h,k). Express your answer in terms of a (crystal period) and b (atom size). It is enough to indicate an expression that is true for all reflexes except for the central one ($h=0$, $k=0$)

Ответ:

$$F(h,k) = \int_0^b \int_0^b e^{iq_x x} e^{iq_y y} dx dy = \frac{e^{iq_x b} - 1}{iq_x} \cdot \frac{e^{iq_y b/2} - e^{-iq_y b/2}}{2iq_x a} \cdot 2a e^{iq_y b/2} =$$
$$= \frac{a^2}{\pi^2} \cdot \left| \frac{\sin(\pi h b/a)}{h} \cdot \frac{\sin(\pi k b/a)}{k} \right| \cdot e^{i\pi b(h+k)/a}$$

Ответ: $|F(h,k)| = \frac{a^2}{\pi^2} \cdot \left| \frac{\sin(\pi h b/a)}{h} \cdot \frac{\sin(\pi k b/a)}{k} \right|, \quad h \neq 0, \quad k \neq 0$

B3 0.60 Look at the diffraction patterns of samples UC1-UC4. For each UC1-UC4 sample, experimentally determine the crystal lattice period a_{UC1}, a_{UC2}, a_{UC3}, a_{UC4}.

Ответ: Periods of samples:

$a_{UC1} = 30\mu\text{m}$

$a_{UC2} = 20\mu\text{m}$

$a_{UC3} = 20\mu\text{m}$

$a_{UC4} = 30\mu\text{m}$

B4 0.40 For each UC1-UC4 sample, find the corresponding crystal structure among Fig. 6. Explain your choice using diagrams, pictures and formulas.

Ответ: Corresponding samples:

UC1 - B
UC2 - A
UC3 - D
UC4 - C

B5 0.80 Determine the size of the atom b.

Ответ: $b = 10\mu m$, estimated like in A5 for UC1.

B6 1.20 Observe the diffraction patterns of samples UC5, UC6, UC7. Determine experimentally the parameters a_1, a_2 and the angle α for each sample. Explain which parameters of the diffraction pattern you are using with the help of diagrams and figures.

Ответ: sample UC5:

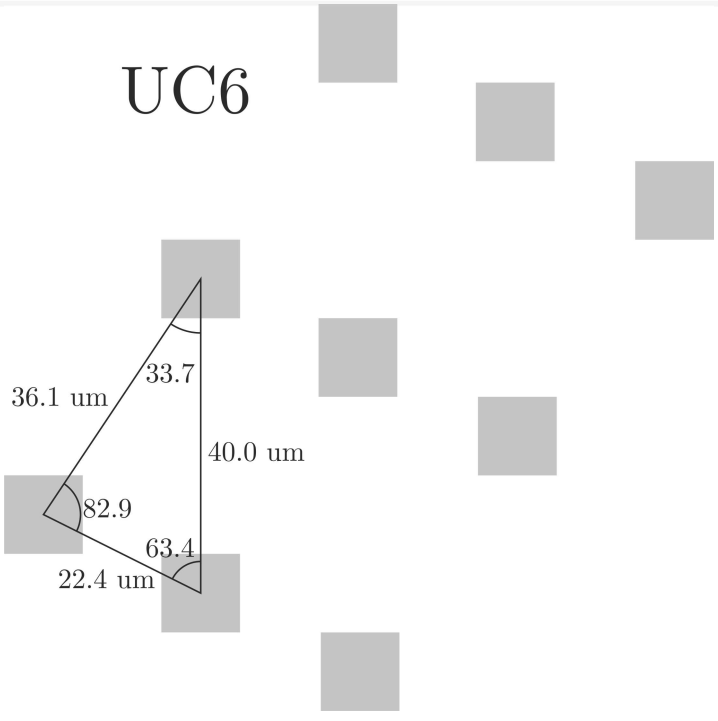
$a_1 = 20\mu\text{m}$, $a_2 = 40\mu\text{m}$, $\alpha = 90^\circ$

sample UC6:

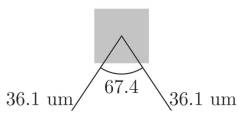
$a_1 = 36.1\mu\text{m}$, $a_2 = 22.4\mu\text{m}$, $\alpha = 63^\circ$

sample UC7:

$a_1 = 40\mu\text{m}$, $a_2 = 36\mu\text{m}$, $\alpha = 56^\circ$



UC7



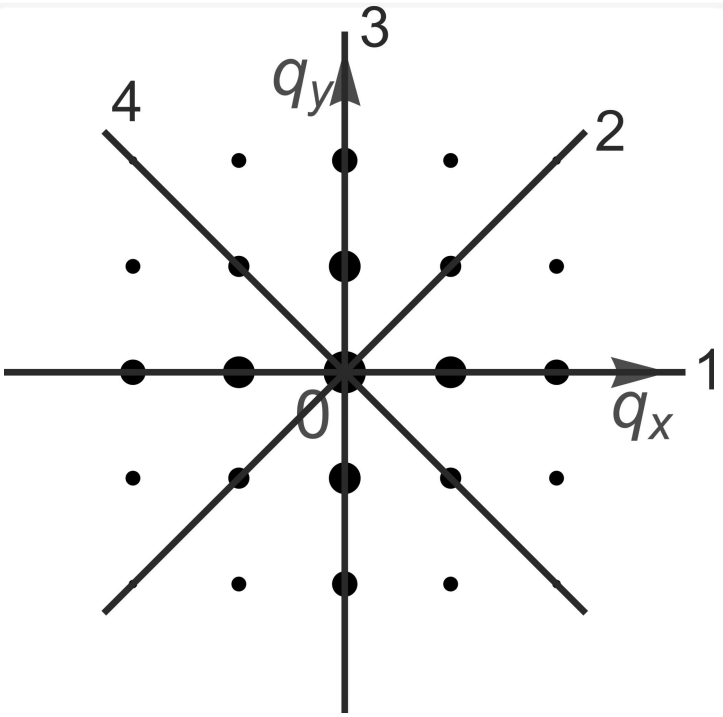
C1 0.30 Specify h and k for the point that will be the center of rotation. What orders of rotational symmetry m are possible for a given image?

Draw all possible axes of mirror symmetry in the image. Name your lines.

Ответ: $h = 0, k = 0$

$m = 1, 2, 4$

All axis of symmetry are presented on figure.



C2 0.20 Specify the equation of the straight line for each axis of mirror symmetry drawn in the previous task. Do not forget to note which equation corresponds to which line.

Ответ: Equations for all possible axis symmetry:

1: $q_y = 0$
2: $q_y = q_x$
3: $q_x = 0$
4: $q_y = -q_x$

C3 0.40 For each rotation symmetry and axis of mirror symmetry, write down the corresponding notation (C_m for rotation and equation for mirror symmetry) and the equation for the intensities $I(q_x, q_y)$, which should take place if this symmetry element is present.

Ответ: Designation (C_m for rotational and equation for axis symmetry) and equation on intensities $I(q_x, q_y)$

C_1: $I(q_x, q_y) = I(q_x, q_y)$

C_2: $I(q_x, q_y) = I(-q_x, -q_y)$

C_4: $I(q_x, q_y) = I(-q_y, q_x)$

$q_y = 0$: $I(q_x, q_y) = I(q_x, -q_y)$

$q_x = 0$: $I(q_x, q_y) = I(-q_x, q_y)$

$q_x = q_y$: $I(q_x, q_y) = I(q_y, q_x)$

$q_x = -q_y$: $I(q_x, q_y) = I(-q_y, -q_x)$

C4 0.20 Write down the equation for the intensities of the reflexes (h, k) and $(-h, -k)$. What symmetry from question C1 corresponds to this equation? Explain your answer.

Structure factor for reflex $-h, -k$:

$$F(-h, -k) = \int \rho(x, y) e^{-i(q_1 hx + q_2 ky)} dxdy = \left(\int \rho(x, y) e^{-i(q_1 hx + q_2 ky)} dxdy \right)^* = F^*(h, k).$$

Here F^* is a complex conjugate of F. This property of $-h, -k$ reflex is only possible because $\rho(x, y)$ is real function.

Intensities are equal

$$I(-h, -k) = F(-h, -k)F^*(-h, -k) = F^*(h, k)F(h, k) = I(h, k).$$

Ответ: Equation on intensities

$I(-h, -k) = I(h, k)$

Which symmetry corresponds? C_2

C5 0.40 Using the definition of the structure factor and symmetry find the structural factors $f_2(q_x, q_y)$, $f_3(q_x, q_y)$, $f_4(q_x, q_y)$ for crystals 2, 3, 4, respectively. Express your answer in terms of the structure factor $F(q_x, q_y) = f_1(q_x, q_y)$ of crystal 1.

Let $\rho(x, y)$ be unit cell for initial crystal. Structure factor for it is:

$$F(q_x, q_y) = \int_{-x0}^{x0} \int_{-y0}^{y0} \rho(x, y) e^{i(q_x x + q_y y)} dxdy,$$

here $x_0 = y_0 = a/2$, a – is a period of lattice. Below limits are presented only when necessary.

For symmetry $x = 0$ unit cell is $\rho_2(x, y) = \rho(-x, y)$, structure factor is

$$f_2(q_x, q_y) = \int_{-x0}^{x0} \int_{-y0}^{y0} \rho_2(x, y) e^{i(q_x x + q_y y)} dxdy = \int_{-x0}^{x0} \int_{-y0}^{y0} \rho(-x, y) e^{i(q_x x + q_y y)} dxdy =$$
$$= \int_{x0}^{-x0} \int_{-y0}^{y0} \rho(x, y) e^{i(-q_x x + q_y y)} d(-x)dy = F(-q_x, q_y).$$

For symmetry $x = y$ unit cell is $\rho_3(x, y) = \rho(y, x)$, structure factor is

$$f_3(q_x, q_y) = \int \rho_3(x, y) e^{i(q_x x + q_y y)} dxdy = \int \rho(y, x) e^{i(q_x x + q_y y)} dxdy =$$
$$= \int \rho(x, y) e^{i(q_y x + q_x y)} dydx = F(q_y, q_x).$$

Point (x, y) moves to new position $(x', y') = (x + x_1, y + y_1)$. That's why $\rho_4(x', y') = \rho(x, y)$.

$$f_4(q_x, q_y) = \int \rho_4(x', y') e^{i(q_x x' + q_y y')} dx'dy' =$$
$$= \left(\int \rho(x, y) e^{i(q_x x + q_y y)} dxdy \right) e^{i(q_x x_1 + q_y y_1)} = F(q_x, q_y) e^{i(q_x x_1 + q_y y_1)}$$

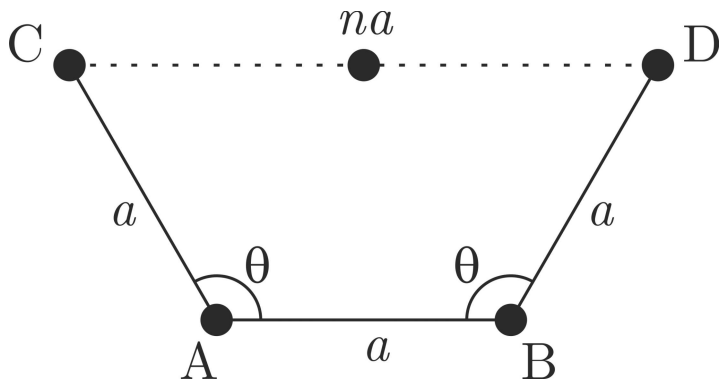
$= \int \int F(x, y) e^{-i(q_x x + q_y y)} dx dy$

Ответ: $f_2(q_x, q_y) = F(-q_x, q_y)$

$f_3(q_x, q_y) = F(q_y, q_x)$

$f_4(q_x, q_y) = F(q_x, q_y) \cdot e^{i(q_x x_1 + q_y y_1)}$

C6 0.50 Consider an arbitrary 2D crystal (Fig. 5). Indicate what orders of m symmetry of rotation can be in 2D crystals. Explain the answer.



Let's consider two nearest atoms A and B (fig.), that means $\vec{AB}$ – lattice vector, $|\vec{AB}| = a$. If rotational symmetry of order m is present, atom C could be obtained from B by rotation by angle $\theta = \frac{2\pi}{m}$. The same way gives atom D from atom A. These two atoms C and D should also be connected by translation symmetry, it means $\vec{CD} = n \cdot \vec{AB}, \quad n \in \mathbb{Z}$.

It leads us to equation

$$\cos\theta = \frac{1-n}{2}.$$

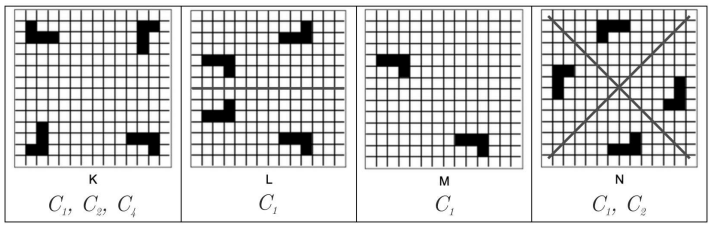
All possible n, $\cos\theta$, θ, m are listed in table.

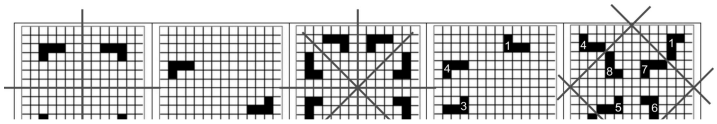
n	$\cos\theta$	θ	m
−1	1	0°	1
0	1/2	60°	6
1	0	90°	4
2	−1/2	120°	3
3	−1	180°	2

Ответ: All possible rotational symmetries: C_1, C_2, C_3, C_4, C_6.

C7 0.90 Determine what symmetries the crystals with unit cells K, L, M, N, and P, Q, R, S, T have. (Fig. in the answer sheet). Draw the axes of mirror symmetry, at the bottom of the picture indicate which rotational symmetries are present on it.

Symmetries of unit cells:





C8 0.80 Observe the diffraction patterns of the samples PG 1, 2, 5, 8. These samples correspond to the unit cells K, L, M, N. Determine what symmetries the given diffraction patterns have. Find the correspondence between patterns and unit cells.

PG	$q_x = 0$	$q_y = 0$	$q_x = q_y$	$q_x = -q_y$	C_4
1	+/- *	+			
2					
5			+	+	
8					+

For PG1 vertical axis of symmetry (q_x=0) must be presented theoretically, but real diffraction pictures could be without that symmetry.

Ответ: One needs to match symmetries of unit cells and diffraction patterns.

PG1 - L
PG2 - M
PG5 - N
PG8 - K

C9 1.00 Observe the diffraction of the samples PG 3, 4, 6, 7, 9. These samples correspond to the unit cells P, Q, R, S, T. Find the correspondence between samples and unit cells. Explain your solution using formulas, diagrams and pictures.

PG	$q_x = 0$	$q_y = 0$	$q_x = q_y$	$q_x = -q_y$	C_4
3	+	+			
4	+	+	+	+	+
6	+	+	+	+	+
7					
9	+	+			

Diffraction patterns from PG4 and PG6 have C_4 symmetry (as unit cells R, T), and all other don't have. In this group symmetries are not enough to match unit cells and diffraction patterns. One needs to use sum rule and to understand that absences are presented for unit cells S and T (as for PG6 and PG9).

PG9. Elements are designated as on fig. Elements 2, 3, 4 were obtained by C_2 symmetry and move.

$$f_1(h,k) = F(h,k).$$

$$f_2(h,k) = F(-h,-k)e^{i5\pi h/7}.$$

$$f_3(h,k) = F(-h,k)e^{-i\pi(2h/7+k)}.$$

$$f_4(h,k) = F(h,-k)e^{-i\pi(h+k)}.$$

For reflexes $(0,k)$

$$f_{PG9}(0,k) = (1+\cos(\pi k))\,(F(0,k)+F(0,-k))\,,$$

each odd $k = 2n + 1, n \in \mathbb{Z}$ is absent. The same is true for another axis $(h, 0)$.

\textbf{PG6.} Let $f_1(h, k) = F(h, k)$. Element 5 could be obtained by mirroring $x = y$ and by moved by $(-a/2, -a/2)$, that's why $f_5(h, k) = F(k, h)e^{-\pi(h+k)}$. Each next pair of elements were obtained by rotating 90°.

Structure factors

$$f_1 + f_5 = F(h, k) + F(k, h)e^{-\pi(h+k)}$$

$$f_2 + f_6 = F(k, -h) + F(h, -k)e^{-\pi(h-k)}$$

$$f_3 + f_7 = F(-h, -k) + F(-k, -h)e^{-\pi(-h-k)}$$

$$f_4 + f_8 = F(-k, h) + F(-h, k)e^{-\pi(-h+k)}$$

For $(0, k)$

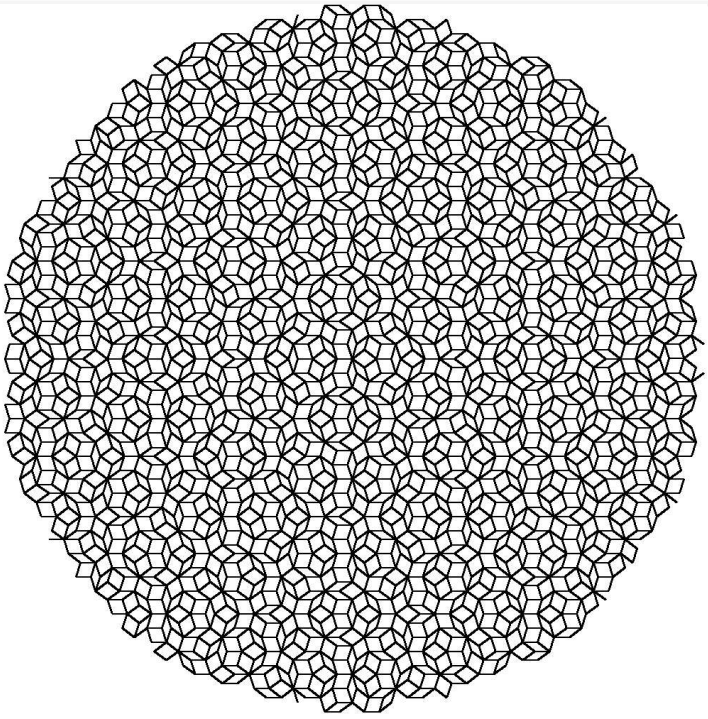
$$f_{PG6}(0, k) = (1 + \cos(\pi k)) \cdot (F(0, k) + F(k, 0) + F(0, -k) + F(-k, 0)),$$

each odd $k = 2n + 1, n \in \mathbb{Z}$ is absent. The same is true for another axis $(h, 0)$.

Ответ: PG3 - P
PG4 - R
PG6 - T
PG7 - Q
PG9 - S

C10 0.30 Observe the diffraction pattern of the UC8 sample. Could this sample be a crystal? Explain your answer.

Ответ: No. Diffraction picture has symmetry C_5. Transnational symmetry for C_5 is not possible.



Structure used as UC8

D1 1.00 The crystal (MR0 or MR2) is illuminated with light with an intensity of I_0. Find the intensity of the maximum at $\vec{q} = 0$.

Electric field is $E_0 = \sqrt{I_0}$ in plane exactly before the diffraction grating. Electric field of the reflex $\vec{q} = 0$ is proportional to transparent area of an unit cell and should be E_0 if the whole unit cell is transparent (i.e. there is no any diffraction grating):

$$E = \frac{N_{transparent}}{N_{all}} E_0.$$

Intensity

$$I = \left(\frac{N_{transparent}}{N_{all}}\right)^2 I_0.$$

Ответ:

$$I_{MR0} = \left(\frac{5}{16}\right)^2 \cdot I_0$$

$$I_{MR2} = \left(\frac{7}{16}\right)^2 \cdot I_0$$

D2 2.00 Determine the structure of the unit cell of the MR1 crystal. The MR1 crystal has one of the indicated units cell (Fig. 12). Describe your solution.

MR1

k \ h	-2	-1	0	1	2
2	5.7	19.1	1.6	24.6	15.8
1	36.5	49.2	40.2	3	20.8
0	2.3	20	336	33.5	0.9
-1	22	3.1	23.2	46.5	25.5
-2	2.9	23.9	2	38.5	7.3

One needs to measure intensities $I(h,k)$ of each reflexes $|h| \leq 2$, $|k| \leq 2$ of MR1.

Transparency of a square with position (χ, γ):

$$\rho(\chi,\gamma) = \sum_{h=-2}^{2} \sum_{k=-2}^{2} \sqrt{I(h,k)} \cdot e^{i\varphi(h,k)} \cdot \exp\left(-2\pi i \left(\frac{\chi}{4}h + \frac{\gamma}{4}k\right)\right),$$

here $\chi \in 0,1,2,3$, $\gamma \in 0,1,2,3$.

Because $\rho(\chi,\gamma) \in \mathbb{R}$, its enough to calculate only real part:

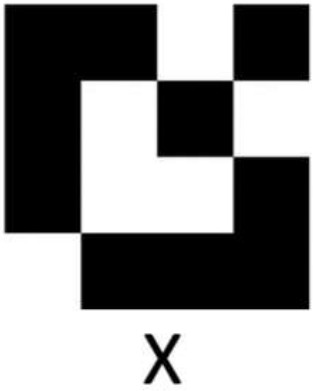
$$\rho(\chi,\gamma) = \sum_{h=-2}^{2} \sum_{k=-2}^{2} \sqrt{I(h,k)} \cdot \cos\left(\varphi(h,k) - \frac{\pi}{2}(\chi h + \gamma k)\right).$$

MR1

γ \ χ	0	1	2	3
3	5.4	4.4	22.6	-9.7
2	-1.7	58.	-13.5	61.1
1	6.	49.1	61.9	-14.4
0	53.8	12.7	-7.5	5.1

Calculated transparencies for MR1 are on figure. It is not necessary to calculate whole 16 squares. One needs to calculate transparency of squares (0,0), (3,3) to understand which values corresponds to transparent and non-transparent values. After that square (3, 2) should be evaluated. It has value close to transparent, that's why structure X is the answer.

Ответ: Unit cell of MR1 - X.



X

D3 2.00 Determine the unit cell structure of the MR2 crystal. The structure of MR2 is similar to MR0: two non-transparent squares have become transparent.

The same formula should be used to find structure of MR2. Intensities of diffraction patterns and calculated transparencies for MR2 are shown on figures

MR2

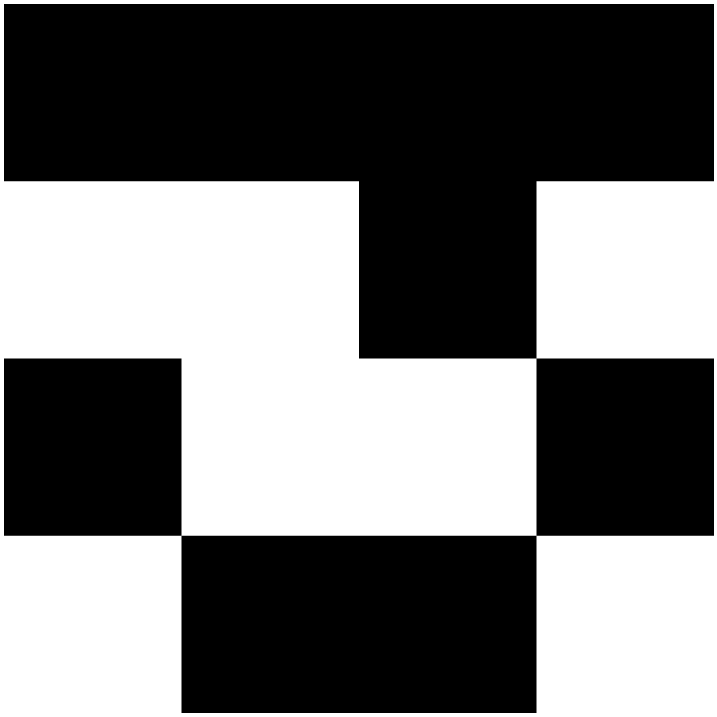
k \ h	-2	-1	0	1	2
2	2	43.3	45.2	39.7	8.9
1	7.2	8.5	69.8	47.9	3.2
0	8.8	11.7	508	4.7	1.7
-1	1.3	40.4	69.7	10	9.3
-2	1.7	56.9	25.8	54.8	1.3

MR2

y \ x	0	1	2	3
3	-10.5	5.8	7.3	-19.4
2	41.5	65.3	2.6	57.9
1	-12.6	46.6	69.1	-0.4
0	54.7	11.9	-5.9	46.7

Because it is known that two non-transparent squares become transparent one needs to calculate transparency only for 11 black squares. Two of them (0, 2) and (3,0) are transparent for MR2.

Ответ: Unit cell of MR2 is on figure:



2020 -- We are what they grow beyond.