两个力学问题 (10 分)

开始题目之前请仔细阅读每个信封中的考试指南。

A 部分. 隐藏的盘子 (3.5 分)

一个半径为 r_1 厚 w_1 的实心木质圆柱，里面固定一个半径为 r_2，厚 w_2 圆柱形金属盘。金属盘对称轴为 B，木质圆柱对称轴为 S，金属盘与木质圆柱同轴放置，S 与 B 之间的距离为 d；金属盘与木质圆柱两底面等距离。木质圆柱的密度为 ρ_1，金属的密度 $\rho_2 > \rho_1$。木质圆柱与金属盘的总质量为 M。

In this task, we place the cylinder on the ground so that it can freely roll to the left and right. See Fig. 1 for a side view and a view from the top of the setup.

在此问中，将木质圆柱横向水平放置在地面上，木质圆柱可以左右自由滚动，见图 1. 侧视图与俯视图。

The goal of this task is to determine the size and the position of the metal disk.

任务的目标是确定金属盘的尺寸 r_2 和 w_2 以及在木质圆柱中的位置 d。

In what follows, when asked to express the result in terms of known quantities, you may always assume that the following are known:

在下面的问题中用已知物理量表示计算结果，并始终假定以下参量为已知，

$$r_1,\ w_1,\ \rho_1,\ \rho_2, M\ . \tag{1}$$

The goal is to determine r2,w2 and d, through indirect measurements.

目标是确定金属盘的尺寸 r_2 和 w_2 以及在木质圆柱中的位置，非直接测量。

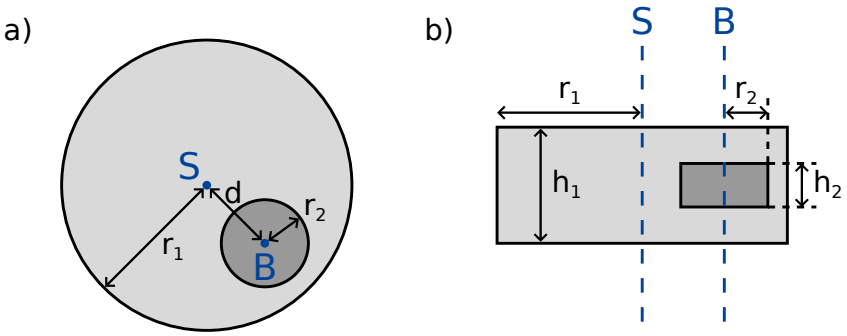


Figure 1: a) side view b) view from top
图 1：a) 侧视图；b) 俯视图

b is the distance between the centre of mass C of the whole system and the symmetry axis S of the cylinder. In order to determine this distance, we design the following experiment: We place the cylinder on a horizontal base in such a way that it is in a stable equilibrium. Let us now slowly incline the base by an angle (see Fig. 2). As a result of the static friction, the cylinder can roll freely without sliding. It will roll down the incline a little bit, but then come to rest in a stable equilibrium after rotating by an angle which we measure.

b 为系统质心 C 与木质圆柱对称轴 S 之间的距离。为确定 b，我们设计了如下实验：将系统放置在一个水平基座上，呈稳定平衡。缓慢倾斜基座使之与地面成 Θ 角 (见图 2)。由于静摩擦力 圆柱与斜面间无滑动滚动。当圆柱沿斜面滚动一小段距离后，再次进入一个稳定平衡状态，测得平衡时圆柱转过的角度为 ϕ。

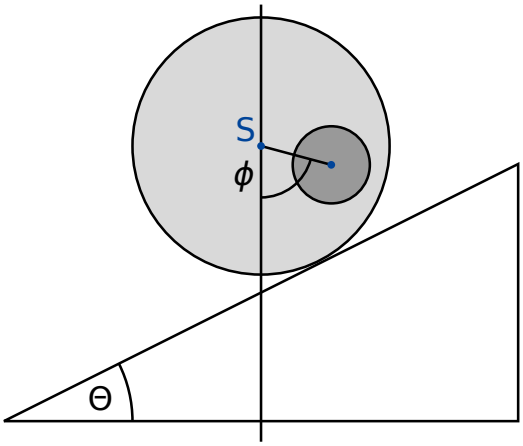


Figure 2: Cylinder on an inclined base.
图 2. 倾斜的基座及圆柱

A.1	Find an expression for b as a function of the quantities (1), the angle and the tilting angle of the base. 给出 b 的表达式，用 (1) 中的参量以及 ϕ 和 Θ 表示。	0.8pt

From now on, we can assume that the value of b is known.

后面的问题，我们可以假定 b 为已知。

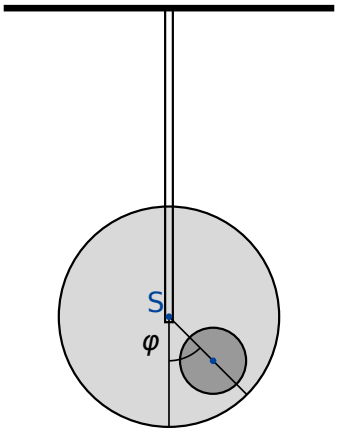


Figure 3: Suspended cylinder.
图 3. 悬挂的圆柱。

Next we want to measure the moment of inertia IS of the cylinder with respect to the symmetry axis S. To this end we suspend the cylinder at its symmetry axis. We then turn it away from its equilibrium position by a small angle , and let it go. See figure 3 for the setup. We find that describes a periodic motion with period T.

接下来我们想确定系统关于 S 轴的转动惯量 I_S。为此 使用硬杆固定 S 轴悬挂圆柱，然后使之相对平衡位置转过一个小角度并释放，如图 3 所示。可以看到系统转角 ϕ 成周期运动，周期为 T。

A.2 What motion does describe? Express the moment of inertia IS of the cylinder around its symmetry axis S in terms of T, b and the known quantities (1). You may assume that we are only disturbing the equilibrium position by a small amount so that is always very small. 0.5pt

给出 ϕ 的运动学方程，指出 ϕ 做何种方式运动? 给出系统关于轴 S 的转动惯量 I_S，用 T, b 和 (1) 中的已知参量表示，假定 ϕ 很小。

From the measurements in questions A.1 and A.2, we now want to determine the geometry and the position of the metal disk inside the cylinder.

通过问题 **A.1** 和 **A.2** 的测量，我们现在进一步确定金属盘的尺寸 r_2 和 w_2 以及在木质圆柱中的位置 d。

A.3 Find an expression for the distance d as a function of band the quantities (1). You may also include r2 and w2 as variables in your expression, as they will be calculated in subtask A.5. 0.4pt

给出 d 的表达式，用 b 和 (1) 中的已知参量表示。表达式中仍可包含变量 r_2 和 w_2，这将在子问题 **A.5** 中进行计算。

A.4 Find an expression for the moment of inertia IS in terms of b and the known quantities (1). You may also include r2 and w2 as variables in your expression, as they will be calculated in subtask A.5. 0.7pt

给出转动惯量 I_S 的表达式，用 b 和 (1) 中已知的参量表示。表达式中仍可包含变量 r_2 和 w_2，这将在子问题 **A.5** 中进行计算。

A.5 Using all the above results, write down an expression for w2 and r2 in terms of b, T and the known quantities (1). You may express w2 as a function of r2. 1.1pt

用以上结果，给出 r_2 和 w_2 的表达式，用 b, T 和 (1) 中已知的参量表示。为书写简单，你可以在 w_2 的表达式中包含 r_2。（建议删去）

B 部分. 旋转的空间站 (6.5 分)

Alice is an astronaut living on a space station. The space station is a gigantic wheel of radius R rotating around its axis, thereby providing artificial gravity to the astronauts. The astronauts live on the inner side of the rim of the wheel. The space station itself is so light that we will neglect its gravitational attraction.

爱丽丝是一个住在空间站的宇航员。空间站是个巨大的半径为 R 的转轮，并关于中轴旋转，由此为宇航员提供了一个假定的重力。宇航员生活在转轮的内边缘。空间站质量很轻，万有引力可忽略；空间站地面的曲率可忽略。

B.1	With what angular frequency ω_{ss} does the space station rotate so that the astronauts experience the same gravity g_E as on the Earth' s surface? 空间站以多大角频率 ω_{ss} 旋转可以使宇航员感受到与在地面相同的重力加速度?	0.5pt

Alice and her astronaut friend Bob have an argument. Bob does not believe that they are in fact living in a space station and claims that they are on Earth. Alice wants to prove to Bob that they are living on a rotating space station by using physics. To this end, she attaches a mass m to a spring with spring constant k and lets it oscillate. The mass oscillates only in the vertical direction, and cannot move in the horizontal direction.

爱丽丝和她的宇航员朋友鲍勃发生争论。鲍勃不相信他们实际生活在空间站，而是生活在地球。爱丽丝想用物理证明给 Bob 他们生活在旋转的空间站。于是，她将质量为 m 的物体挂在弹性系数为 k 的弹簧上，并让其振动。重物只在垂直方向振动，在水平方向上无运动。

B.2	Assuming that on Earth gravity is constant with acceleration g_E, what would be the angular oscillation frequency ω_E that one measures? 在地球上实验，假定重力加速度 g_E 是恒定的，地球上的人可测得上述振动的角频率 ω_E 为多大?	0.2pt

B.3	What angular oscillation frequency ω does Alice measure on the space station? 爱丽丝在空间站 测得的角频率 ω 为多大?	0.6pt

Alice is convinced that her experiment proves that they are on a rotating space station. Bob remains skeptical. He claims that when taking into account the change in gravity above the surface of the Earth, one finds a similar effect. Is he right?

爱丽丝确信她的实验结果能够证明他们生活在一个旋转的空间站。鲍勃仍保持怀疑，他声称当考虑地球表面以外引力随空间的变化，仍可得到类似现象。接下来我们探究为何鲍勃是对的。

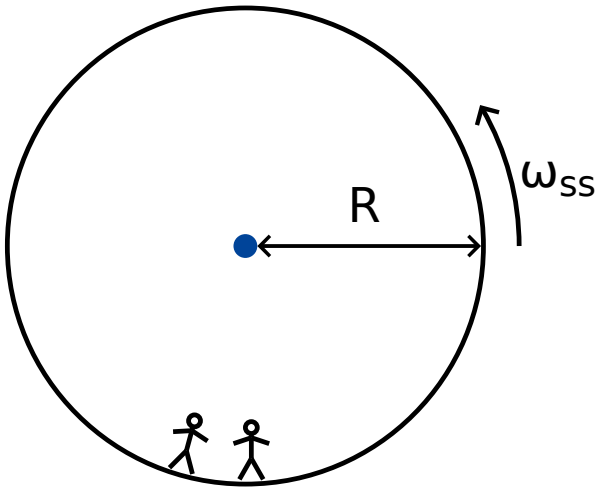


图 4. 空间站

B.4 Derive an expression of the gravity $g_E(h)$ for small heights h above the surface of the Earth and compute the angular frequency $\tilde{\omega}_E$ (linear approximation is enough). The radius of the Earth is given by R_E. 0.8pt
推导地球表面附近高度 h 为小量处重力加速度 $g_E(h)$ 的表达式，并计算振动频率 $\tilde{\omega}_E$ (线性近似足矣)。已知地球半径为 R_E。忽略地球自转。

Indeed, Alice finds that the spring pendulum oscillates with the frequency that Bob predicted.

事实上，爱丽丝确实发现对于该空间站弹簧的振动与 鲍勃所指出的频率一致。

B.5 For what radius R of the space station does the oscillation frequency ω match the oscillation frequency $\tilde{\omega}_E$ on the surface of the Earth? 0.3pt
空间站的半径 R 为多大 ω 和 $\tilde{\omega}_E$ 相一致？结果用 R_E 表示。

Exasperated with Bob’ s stubbornness, Alice comes up with the idea of using the Coriolis force to prove her point. To this end she climbs on a tower of height H over the ground of the space station and drops a mass.

被鲍勃的顽固所激怒，爱丽丝想到了一个实验证明她的观点。为此她爬到距空间站地面高为 H 的塔顶并下落了一个物体。这个实验可以理解为在一转动参照系中，也可以理解为在一惯性参照系中。

The Coriolis force is a fictitious force that appears in a uniformly rotating frame of reference. The force $\vec{F}_C$ acting on an object of mass m moving at velocity $\vec{v}$ in a rotating frame with constant angular frequency $\vec{\omega}_{ss}$ is given by

在一匀速转动参照系中，宇航员会感受到一个称作科里奥利力 $\vec{F}_C$ 的假想力。在角速度为 $\vec{\omega}_{ss}$ 转动参照系中，作用于质量为 m，以速度 $\vec{v}$ 运动的物体，科里奥利力 $\vec{F}_C$ 表示为：

$$\vec{F}_C = 2m\vec{v} \times \vec{\omega}_{ss} \,. \tag{2}$$

In terms of the scalar quantities you may use

也可以标量表示为：

$$F_C = 2mv\omega_{ss} \sin\phi \,, \tag{3}$$

where ϕ is the angle between the velocity and the axis of rotation. The force is perpendicular to both the velocity v and the axis of rotation. The sign of the force can be determined from the right-hand rule, but in what follows you may choose it freely.

其中 ϕ 为速度与转轴的夹角，科里奥利力与速度 v 和转轴垂直。力的符号由右手定则确定。but in what follows you may choose it freely.（建议删去不译）

B.6 计算物体落到地面时的水平方向速度 v_x 和位移 d_x（相对于塔基，在垂直于塔的方向上），假定 H 很小，下落过程中加速度不变，并假定 $d_x \ll H$ 1.1pt

To get a good result, Alice decides to conduct this experiment from a much taller tower than before. To her surprise, the mass hits the ground at the base of the tower, so that $d = 0$.

为得到一个更好的结果，爱丽丝决定在一个更高的塔实施实验，让她惊奇的是重物落在了塔基处，即 $d_x = 0$。

B.7 计算出发生上述现象（即，$d_x = 0$）塔高的下限。 1.3pt

Alice is willing to make one last attempt at convincing Bob. She wants to use her spring oscillator to show the effect of the Coriolis force. To this end she changes the original setup: She attaches her spring to a ring which can slide freely on a horizontal rod in the x direction without any friction. The spring itself oscillates in the y direction. The rod is parallel to the ground and perpendicular to the axis of rotation of the space station. The xy plane is thus perpendicular to the axis of rotation, with the y direction pointing straight towards the center of rotation of the station.

爱丽丝想为说服鲍勃做最后努力。她想用她的弹簧振子演示科里奥利力的效应。于是她改变了原来的实验装置：她将弹簧挂在一个环上，环可以在沿 x 方向的水平杆上无摩擦自由滑动。弹簧本身在 y 方向振动。杆平行于空间站地面，垂直于空间站转轴。因此 xy 平面也垂直于空间站转轴，y 方向指向空间站转动中心。

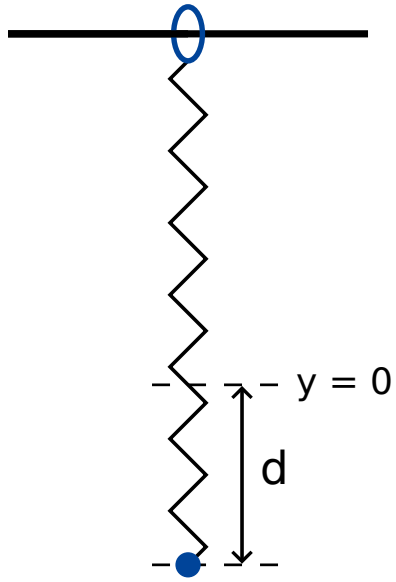


Figure 4: Setup.
图 5. 装置

B.8 Alice pulls the mass a distance d downwards from the equilibrium point $x = 0$, $y = 0$, and then lets it go (see figure 4). 1.7pt
爱丽丝向下方拉动重物离开平衡点（$x = 0, y = 0$）距离为 d，然后释放（见图 5）。

- Give an algebraic expression of $x(t)$ and $y(t)$. You may assume that $\omega_{ss} d$ is small.
  给出 $x(t)$ 和 $y(t)$ 的代数表达式，可假定 $\omega_{ss} d$ 很小，并忽略 y 方向上运动引起的科里奥利力。
- Sketch the trajectory $(x(t), y(t))$, marking all important features such as amplitude.
  画出运动轨迹 $(x(t), y(t))$。标示出所有重要特征，如振幅等。

Nonlinear Dynamics in Electric Circuits (10 points) 电路中的非线性力学 (10 分)

Please read the general instructions in the separate envelope before you start this problem.

开始题目之前请仔细阅读另一信封中总指引

介绍

Bistable non-linear semiconducting elements (e.g. thyristors) are widely used in electronics as switches and generators of electromagnetic oscillations. The primary field of applications of thyristors is controlling alternating currents in power electronics, for instance rectification of AC current to DC at the megawatt scale. Bistable elements may also serve as model systems for self-organization phenomena in physics (this topic is covered in part B of the problem), biology (see part C) and other fields of modern nonlinear science.

双稳态非线性半导体元件（例如晶闸管 thyristors）广泛用于电子开关和电磁振荡发生器。晶闸管应用的主要领域是在电力电子产品中控制交流电流，例如，将兆瓦级别交流电整流为直流电。双稳态元件也可以作为模型系统模拟物理学中自组织现象（B 部分讨论的就是这个问题），生物学（见 C 部分）和现代非线性科学等领域。

目标

To study instabilities and nontrivial dynamics of circuits including elements with non-linear $I-V$ characteristics. To discover possible applications of such circuits in engineering and in modeling of biological systems.

研究包含非线性 I-V 特性元件的电路的不稳定性和复杂动力学。探究此类电路在工程和生物系统建模中的应用。

Part A. Stationary states and instabilities (3 points) A 部分 定态与不稳定性（3 分）

Fig. 1 shows the so-called **S-shaped** $I-V$ characteristics of a non-linear element X. In the voltage range between $U_{\mathrm{h}}=4.00\ \mathrm{V}$ (the holding voltage) and $U_{\mathrm{th}}=10.0\ \mathrm{V}$ (the threshold voltage) this $I-V$ characteristics is multivalued. For simplicity, the graph on Fig. 1 is chosen to be piece-wise linear (each branch is a segment of a straight line). In particular, the line in the upper branch touches the origin if it is extended. This approximation gives a good description of real thyristors.

图（1）所示的是某非线性元件 X 具有的所谓 S-形 I-V 特性。在电压范围 $U_{\mathrm{h}}=4.00\ \mathrm{V}$（保持电压）和 $U_{\mathrm{th}}=10.0\ \mathrm{V}$（阈值电压）间，此 I-V 特性电压是多值的。为简单起见，图（1）中的图线**(曲线)** 可被视为分段线性（每个分段是直线的一段）。**特别的是，上段分支的直线如延伸将触碰原点（上段分支直线如延伸将与原点相交）**。这种近似是对真实晶闸管的一个相当好的描述。

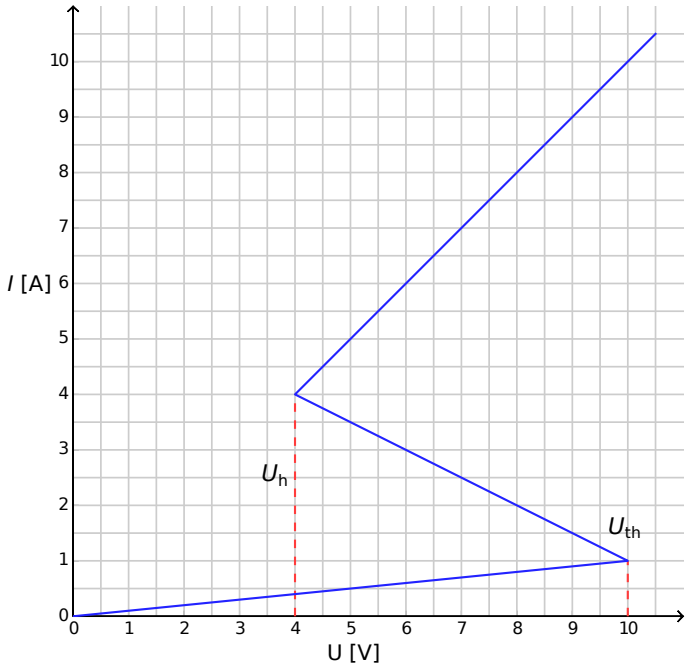


Figure 1: I　Vcharacteristics of the non-linear element X.
图（1）：非线性元件 X 的 $I-V$ 特性图（**特性曲线**）

A.1 Using the graph, determine the resistance R_{on} of the element X on the upper branch of the $I-V$ characteristics, and R_{off} on the lower branch, respectively. The middle branch is described by the equation 0.4pt
使用图线（依据图示曲线），分别计算 X 元件 I-V 特性图线**(曲线)** 上方分段的电阻值 R_{on}，下方分段的电阻值 R_{off}。中间分段的电阻 R_{int} 满足

$$I = I_0 - \frac{U}{R_{\text{int}}}. \tag{1}$$

Find the values of the parameters I_0 and R_{int}.
计算参数 I_0 和 R_{int} 的值

The element X is connected in series (see Fig.2) with a resistor R, an inductor L and an ideal voltage source $\mathcal{E}$. One says that the circuit is in a stationary state if the current is constant in time, $I(t) = \text{const}$.

如图（2）元件 X 和电阻 R 电感 L 及一个理想电压源 $\mathcal{E}$ 串联在一起，所述（**当**）电路的电流不随时间而变，即 $I(t) =$　常数 时，我们称此电路处于定态**(稳恒态)**。

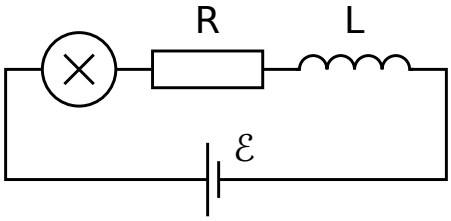


Figure 2: Circuit with element X, resistor R, inductor L and voltage source E.
图（2）：元件 X、电阻 R、电感 L 和电压源 $\mathcal{E}$ 的电路

A.2 What are the possible numbers of stationary states that the circuit of Fig. 2 may have for a fixed value of $\mathcal{E}$ and for $R = 3.00\,\Omega$? How does the answer change for $R = 1.00\,\Omega$? 1pt
图（2）电路中当 $\mathcal{E}$ 值不变，$R = 3.00\,\Omega$ 时，可能的定态**(稳恒态)** 数量是几个，当 $R = 1.00\,\Omega$ 时的答案是多少？

A.3 Let $R = 3.00\,\Omega, L = 1.00\,\mu\mathrm{H}$ and $\mathcal{E} = 15.0\,\mathrm{V}$ in the circuit shown in Fig. 2. Determine the values of the current $I_{\mathrm{stationary}}$ and the voltage $V_{\mathrm{stationary}}$ on the non-linear element X in the stationary state. 0.6pt
在图（2）中设 $R = 3.00\,\Omega, L = 1.00\,\mu\mathrm{H}$ 和 $\mathcal{E} = 15.0\,\mathrm{V}$，计算定态时非线性元件 X 的电流 $I_{\mathrm{stationary}}$ 值和电压 $V_{\mathrm{stationary}}$ 值。

The circuit in Fig. 2 is in the stationary state with $I(t) = I_{\mathrm{stationary}}$. This stationary state is said to be stable if after a small displacement (increase or decrease in the current), the current returns towards the stationary state. And if the system keeps moving away from the stationary state, it is said to be unstable.

当图(2)处于电流 $I(t) = I_{\mathrm{stationary}}$ 的定态下，如果在一个小位移 (电流增加或减少) 之后, 电流返回到定态，这种定态是稳定的。

A.4 Use numerical values of the question **A.3** and study the stability of the stationary state with $I(t) = I_{\mathrm{stationary}}$. Is it stable or unstable? 1pt
使用 A.3 小题的数值结果，研究 $I(t) = I_{\mathrm{stationary}}$ 的定态的稳定性。此定态是稳定的还是不稳定的?

Part B. Bistable non-linear elements in physics: radio transmitter (5 points) B 部分 双稳态非线性元件物理应用：无线电发射器（5 分）

We now investigate a new circuit configuration (see Fig. 3). This time, the non-linear element X is connected in parallel to a capacitor of capacitance $C = 1.00\,\mu\mathrm{F}$. This block is then connected in series to a resistor of resistance $R = 3.00\,\Omega$ and an ideal constant voltage source of voltage $\mathcal{E} = 15.0\,\mathrm{V}$. It turns out that this circuit undergoes oscillations with the non-linear element X jumping from one branch of the $I - V$ characteristics to another over the course of one cycle.

现在我们研究一个新的电路配置（参见图 3)。此时，非线性元件 X 与电容 $C = 1.00\,\mu\mathrm{F}$ 的电容并联。然后该模块与 $R = 3.00\,\Omega$ 的电阻和 $\mathcal{E} = 15.0\,\mathrm{V}$ 的理想恒压源串联。事实证明这个电路会发生振荡，其非线性元件 X 在一个周期内会从 $I - V$ 特性的一个分段跳变到另一个分段。

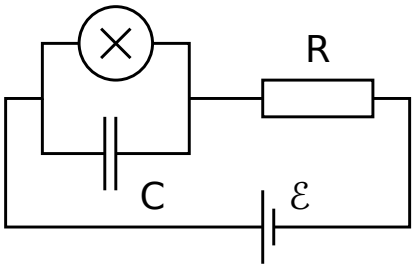


Figure 3: Circuit with element X, capacitor C, resistor R and voltage source E.
图（3）：包含元件 X、电容 C、电阻 R 和电压源 $\mathcal{E}$ 的电路。

B.1 Draw the oscillation cycle on the $I-V$ graph, including its direction (clockwise or anticlockwise). Justify your answer with equations and sketches. 1.8pt
在 $I-V$ 图上绘制振荡周期图，包括它的方向（顺时针或逆时针）。并使用方程和简图证明你的答案。

B.2 Find expressions for the times t_1 and t_2 that the system spends on each branch of the $I-V$ graph during the oscillation cycle. Determine their numerical values. Find the numerical value of the oscillation period T assuming that the time needed for jumps between the branches of the $I-V$ graph is negligible. 1.9pt
时间 t_1 和 t_2 为一个振荡周期内 $I-V$ 图线每个分段的时间，求 t_1 和 t_2 的表达式，求出它们的数值。假设 I-V 图的不同分段间跳跃所需时间忽略不计，求出振荡周期 T 的数值。

B.3 Estimate the average power P dissipated by the non-linear element over the course of one oscillation. An order of magnitude is sufficient. 0.7pt
估计**(估算)** 一个振荡周期非线性元件消耗的平均功率 P。给出其数量级。

The circuit in Fig. 3 is used to build a radio transmitter. For this purpose, the element X is attached to one end of a linear antenna (a long straight wire) of length s. The other end of the wire is free. In the antenna, an electromagnetic standing wave is formed. The speed of electromagnetic waves along the antenna is the same as in vacuum. The transmitter is using the main harmonic of the system, which has period T of question **B.2**.

图(3)中的电路被用作无线电发射器。为了这个目的(**出于该用途**)，元件 X 被连接到长度为 s 的线性天线(长直导线)的一端。**导线的另一端是自由的。在天线上形成一个电磁驻波。沿着天线的电磁波的速度和真空中是一样的。发射器是（没有**

B.4 What is the optimal value of s assuming that it cannot exceed $1\ \mathrm{km}$? 0.6pt
s**(去掉 s)** 在不超过 1km 的情况下，s 的最佳值是多少？

Part C. Bistable non-linear elements in biology: neuristor (2 points) C 部分。双稳态非线性元件生物学应用：类神经器件（2 分）

In this part of the problem, we consider an application of bistable non-linear elements to modeling of biological processes. A neuron in a human brain has the following property: when excited by an external signal, it makes one single oscillation and then returns to its initial state. This feature is called excitabil-

ity. Due to this property, pulses can propagate in the network of coupled neurons constituting the nerve systems. A semiconductor chip designed to mimic excitability and pulse propagation is called a neuristor (from neuron and transistor).

在该 (没有该) 问题的这一部分，我们考虑 双稳态非线性元件的生物过程的建模应用**(在生物过程建模中的应用)**。人脑中的一个神经元具有以下特性：当受外部信号刺激时，它产生一个单一振荡，然后返回到它的初始状态。此特性被称为应激性。由于这种特性，脉冲可以在 耦合神经元网络中传播，这便构建了神经系统。被设计成模拟应激性和脉冲传播的半导体芯片称为类神经器件（neuristor） （neuristor 由英文中神经元和晶体管两个单词衍生）

We attempt to model a simple neuristor using a circuit that includes the non-linear element X that we investigated previously. To this end, the voltage $\mathcal{E}$ in the circuit of Fig. 3 is decreased to the value $\mathcal{E}' = 12.0\ \mathrm{V}$. The oscillations stop, and the system reaches its stationary state. Then, the voltage is rapidly increased back to the value $\mathcal{E} = 15.0\ \mathrm{V}$, and after a period of time τ (with $\tau < T$) is set again to the value $\mathcal{E}'$ (see Fig. 4). It turns out that there is a certain critical value $\tau_{\mathrm{crit.}}$, and the system shows qualitatively different behavior for $\tau < \tau_{\mathrm{crit}}$ and for $\tau > \tau_{\mathrm{crit}}$.

我们尝试使用我们先前研究的包含非线性元素 X 的电路来模拟类神经器件（neuristor)。为此，将图（3）电路中的电压 $\mathcal{E}$ 减小到值 $\mathcal{E}' = 12.0\ \mathrm{V}$. 此时振荡停止，并且系统达到它的定态。然后，电压迅速增加回升至 $\mathcal{E} = 15.0\ \mathrm{V}$，并在经过一段时间 τ 后（且 $\tau < T$）被再次设置到值 $\mathcal{E}'$（如图 4 所示)。事实证明，存在一个时间临界值 $\tau_{\mathrm{crit.}}$，当 $\tau < \tau_{\mathrm{crit}}$ 和 $\tau > \tau_{\mathrm{crit}}$ 时系统的行为完全不同。

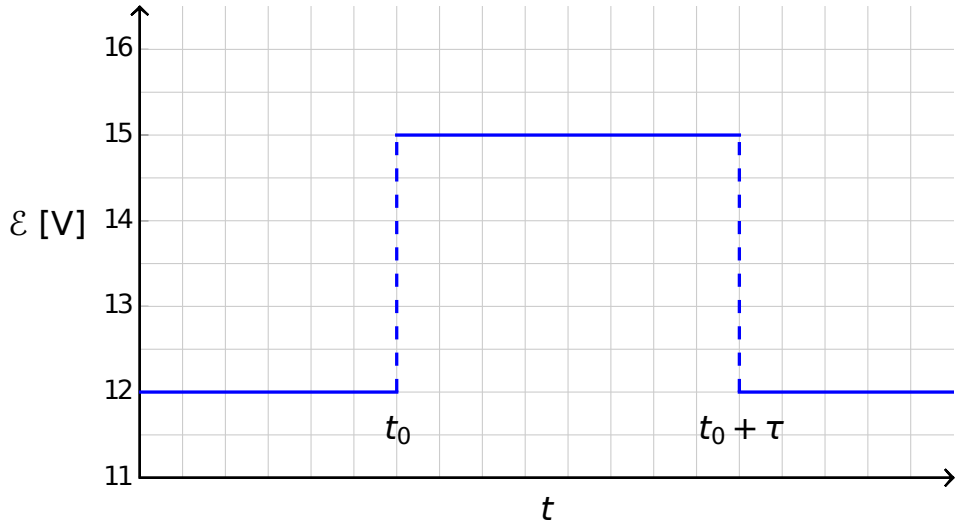


Figure 4: Voltage of the voltage source as a function of time.
图（4)：电压源的电压随时间变化

C.1	Sketch the graphs of the time dependence of the current $I_X(t)$ on the non-linear element X for $\tau < \tau_{\mathrm{crit}}$ and for $\tau > \tau_{\mathrm{crit}}$. 绘制非线性元件 X 的电流 $I_X(t)$ 的时间变化图，当 $\tau < \tau_{\mathrm{crit}}$ 和 $\tau > \tau_{\mathrm{crit}}$ 时分别画图**(分 $\tau < \tau_{\mathrm{crit}}$ 和 $\tau > \tau_{\mathrm{crit}}$ 两种情况分别画图)**。	1.2pt

C.2	Find the expression and the numerical value of the critical time τ_{crit} for which the scenario switches. 求出场景切换的临界时间 τ_{crit} 的表达式，并计算其数值。	0.6pt

C.3 Is the circuit with $\tau = 1.00 \times 10^{-6}\,\mathrm{s}$ a neuristor? 0.2pt
对于以上电路，当 $\tau = 1.00 \times 10^{-6}\,\mathrm{s}$ 时，还能模拟一个类神经器件吗?

大型强子对撞机（10 分）

在开始答题前，请阅读另外一个信封中的理论考试指南。

本题讨论在欧洲核子研究中心（CERN）的粒子加速器：LHC（大型强子对撞机）中的物理问题。CERN 是世界上最大的粒子物理实验室。它的主要研究目标是观测自然的基本规律。通过强磁场下的加速器圆环，两束粒子加速到很高的能量，然后相互碰撞。粒子在加速器圆周上的分布虽然是不均匀的, 但是它们会聚集形成所谓的粒子束。由此碰撞而产生的新的粒子，可以通过一些大型探测器进行观测。表 1 给出了 LHC 的一些参数。

LHC 圆环	
圆环的周长	$26659\ \mathrm{m}$
质子流包含的束数	2808
每一束包含的质子数目	1.15×10^{11}
质子流	
质子的能量	$7.00\ \mathrm{TeV}$
质心能量	$14.0\ \mathrm{TeV}$

表 1: LHC 相关参数的典型值

粒子物理学家对能量、动量和质量采取简便的单位: 能量以电子伏特【eV】为单位。根据定义，$1\ \mathrm{eV}$ 是一个带有基本电荷电量 e 的粒子穿过一伏特的电位差后所获得的能量。（$1\ \mathrm{eV} = 1.602 \cdot 10^{-19}\ \mathrm{kg\,m^2/s^2}$。）

动量以 eV/c 为单位，质量以 eV/c^2 为单位，这里 c 是真空中的光速。因为 1ev 是相当小的能量，粒子物理学家通常使用 MeV（$1\ \mathrm{MeV} = 10^6\ \mathrm{eV}$）、$\mathrm{GeV}$（$1\ \mathrm{GeV} = 10^9\ \mathrm{eV}$）和 TeV（$1\ \mathrm{TeV} = 10^{12}\ \mathrm{eV}$）。

题目的 A 部分研究质子或电子的加速。B 部分关注于如何识别欧洲核子研究中心碰撞产生的各种粒子。

A 部分. LHC 加速器 (6 分)

加速:

假设质子在加速电压 V 下被加速到速度非常接近光速，忽略由于辐射和与其他粒子碰撞引起的任何能量损失。

A.1 求质子的末速度 v 的确切表达式，以加速电压 V 和一些物理常量表示。 0.7pt

设计一个未来在欧洲核子研究中心的实验计划，利用 LHC 产生的质子与能量为 $60.0\ \mathrm{GeV}$ 的电子碰撞。

A.2 对于具有高速能量和低质量的粒子，末速度 v 与光速的相对偏差 $\Delta = (c - v)/c$ 是很小的。求出 Δ 的一阶近似表达式，用加速电压 V 和一些物理常数来表示。对于能量为 $60.0\ \mathrm{GeV}$ 电子，计算 Δ 的值。 0.8pt

我们现在接着研究 LHC 中的质子。假设质子束流的截面是圆形。

A.3 要使质子束在保持在圆形轨道上，求出均匀磁感应强度 B 的表达式。表达式应该只包含质子的能量 E, 周长 L, 一些基本常数和数字。如果能保证对计算结果的影响小于有效数字的最少位数物理量所给出的精度，你可以采取适当的近似。
要获得 $E = 7.00\ \mathrm{TeV}$ 能量的质子，求出均匀磁感应强度 B 的值，计算中忽略质子间的相互碰撞。 1.0pt

辐射功率

加速带电粒子以电磁波的形式向外辐射能量。一个具有恒定角速度做圆周运动的带电粒子，其辐射功率 P_{rad} 仅仅只取决于加速度 a, 所带电荷 q, 光速 c 和自由空间的介电常数 ε_0。

A.4	用量纲分析法，求出辐射功率 P_{rad} 的表达式。	1.0pt

真实的辐射功率公式还包含一个因子 $1/(6\pi)$; 并且由全相对论可以推导出，还需要再乘以一个额外因子 γ^4， 这里 $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$.。

A.5	LHC 的一个质子的能量 $E = 7.00\ \mathrm{TeV}$(在表 1 中已给出)，计算这个质子的总辐射功率 P_{tot} 的值，计算中可以使用适当的近似。	1.0pt

直线加速:

在 CERN，质子由静止状态，经过长度 $d = 30.0\ \mathrm{m}$，电位差 V = 500 MV 的直线加速器加速。假定加速电场是均匀的，这个线性加速器由两个平板组成，图 1 为其示意图。

A.6	求出质子通过这个场的时间 T 的数值。	1.5pt

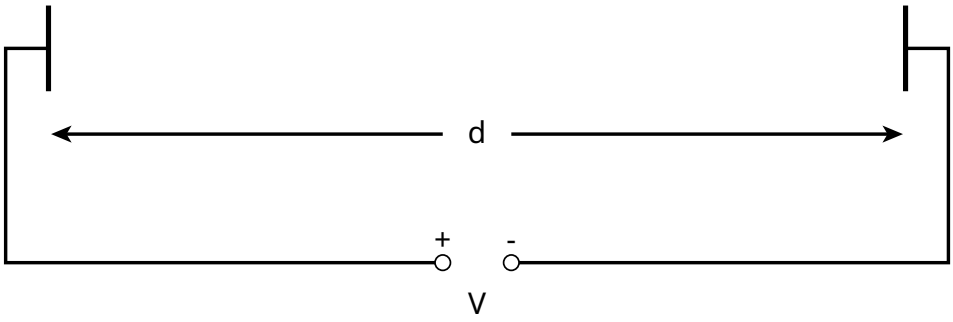


图 1: 加速器模块示意图。

B 部分. 粒子的识别 (4 分)

飞行时间

对碰撞中生成的高能粒子进行识别，对解释碰撞相互作用过程是很重要的。识别粒子的一个简单的方法，是测量一个已知动量的粒子通过长度为 l 的一种叫做飞行时间 (ToF) 探测器所需要的时间 (t)。通过探测器的典型粒子的质量见表 2。

粒子	**质量** [MeV/c^2]
氘核	1876
质子	938
带电 K 介子	494
带电 π 介子	140
电子	0.511

表 2: 各种粒子及其质量.

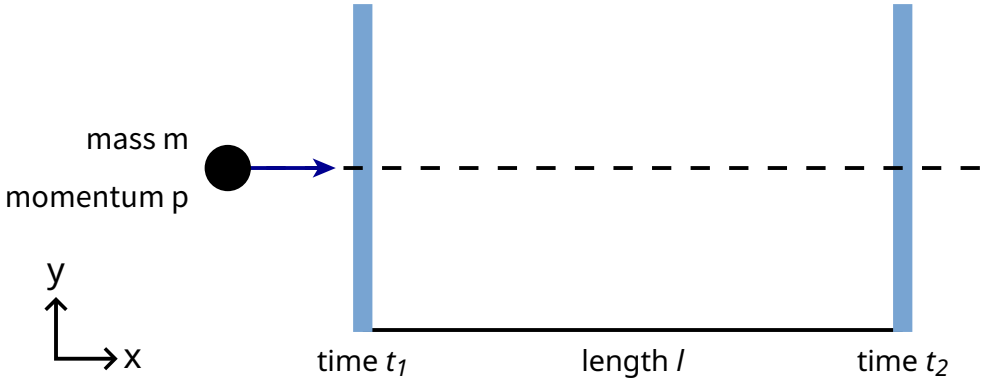
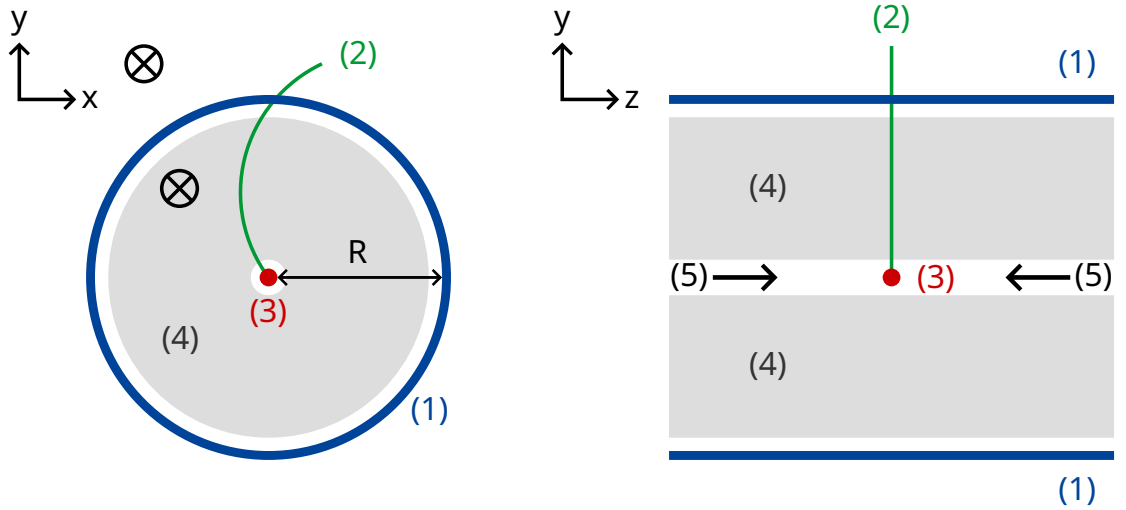


图 2: 飞行时间 (ToF) 探测器结构示意图，图中英文含义：mass 质量，momentum 动量，time 时间，length 长度

B.1 假设带有基本电荷电量 e 的粒子以接近光速 c 的速度，沿直线匀速通过 ToF 探测器，并且飞行的直线轨迹与探测器的两个探测平面垂直 (见图 2)。求粒子的静止质量 m 的函数表达式，用动量 p、飞行长度 l 和飞行时间 t 表示。 0.8pt

B.2 要准确分辨动量均为 1.00 GeV / c 的带电 k 介子和带电 π 介子，计算 ToF 探测器所需的最小长度 l。为了很好区分这两种粒子，要求飞行时间的差异大于 ToF 探测器分辨时间的三倍。对于 ToF 探测器，分辨时间的典型值是 $150\ \mathrm{ps}\ (1\ \mathrm{ps} = 10^{-12}\ \mathrm{s})$。 0.7pt

在后面的题目中，对产生粒子进行探测的典型 LHC 探测器分为两个：一个是轨迹探测器，另一个是 ToF 探测器。图 3 在与质子束流平行的和垂直的两个平面内展示了装置结构。两个探测器是位于相互作用区的一些管子，粒子束流通过这些管子的中心。轨迹探测器测量带电粒子通过磁场时的运行轨迹，磁场的方向平行于质子束流的方向。通过运行轨道的半径 r 可以确定粒子的横向动量 $\mathrm{p_T}$。因为碰撞时间已知，ToF 探测器只需要用一个管子进行飞行时间测量 (从碰撞到被 ToF 管探测到所经过的时长)。这个 ToF 管位于轨道探测器后面。在本任务中，你可以假设所有由碰撞产生的粒子，其运动方向垂直于质子束流，这意味着新生成的粒子没有沿着质子流方向的纵向动量。



transverse plane

cross section of the longitudinal view at the center of the tube along the beamline

(1) - ToF 管
(2) - 轨迹
(3) - 碰撞点
(4) - 轨迹探测管
(5) - 质子流
⊗- 磁场
图 3 : 具有一个轨迹探测器和一个 TOF 探测器的粒子探测实验装置结构图。两个探测器是以碰撞区为中心分布的一些管子。左图: 与束流方向垂直的截面上的横向示意图；右图: 与束流方向平行的截面上的纵向示意图。
图中英文含义：transverse plane 横向垂直平面，cross section of the longitudinal view at the center along the beam line of the tube 沿束流方向的过探测管中心的纵向视图

B.3 求粒子质量表达式，用磁感应强度 B, ToF 管半径 R，基本常数，轨迹半径 r 和飞行时间 t（后两个物理量可以测量得到）来表示。 1.7pt

我们探测到四种粒子，现在要对它们进行辨别。轨迹探测器的磁感应强度为 $B = 0.500\ \mathrm{T}$。ToF 管的半径 R 为 $3.70\ \mathrm{m}$。以下是测量结果（$1\ \mathrm{ns} = 10^{-9}\ \mathrm{s}$）：

粒子	轨迹半径 r [m]	飞行时间 t [ns]
A	5.10	20
B	2.94	14
C	6.06	18
D	2.31	25

B.4 通过对粒子质量的计算，分辨出这四种粒子。 0.8pt

二维材料的电导率

在开始进行本实验之前，请认真阅读在另外一个单独信封里装着的“总实验指南”。

引言

在研发基于半导体技术的新一代元器件（比如计算机芯片、太阳能电池）时，研究人员一直在寻找具有优越输运特性 (如低电阻率) 的材料。这些特性的测量使用的是有限尺寸的样本进行的，且测量时的接触电阻有限，样品具有一个特殊的几何结构。在对这些测量结果进行分析，以得到材料的真正特性时，必须考虑到样品有限尺寸等上述因素的影响。此外，材料在薄膜状态下的特性可能不同于材料处于块状下的特性。

在本实验题中，我们将测量材料的电学特性。测量中将使用下面两种定义：

- 电阻 R: 电阻是样品或元件的一种电学性质。对于具有特定尺寸的给定样品，可以测量出其电阻值。
- 电阻率ρ: 电阻率是用来确定电阻的材料属性。它取决于材料本身参数和外部参数，如温度，但它不依赖于样品的几何形状。

特别地，我们将要测量所谓的薄片电阻率。这个电阻率是上面提到的电阻率除以薄层的厚度。

我们将探讨下列参数对薄层材料的电阻测量所带来的影响：

- 测量电路
- 几何结构
- 以及样品尺寸

实验样品有：一片导体纸，镀有金属膜的硅晶片。

材料清单：

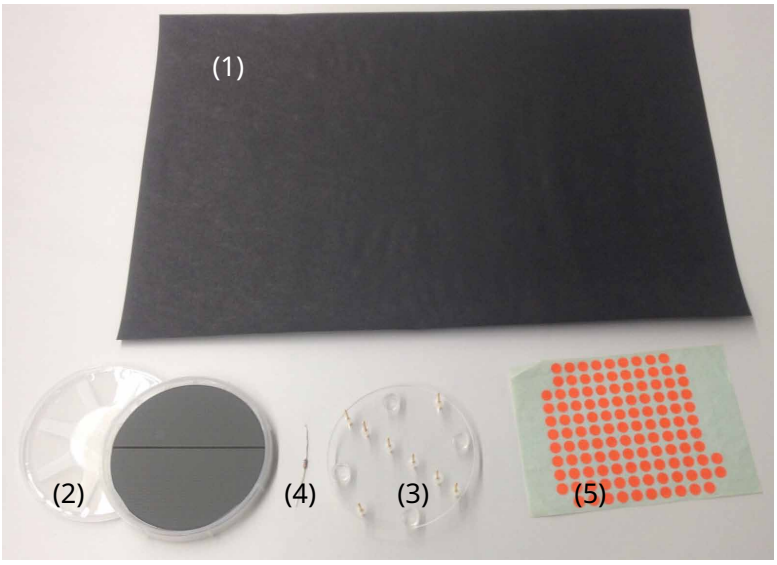


图 1：本实验中需要使用的其他元件

1、涂有一层石墨的导电纸

2、镀有铬薄膜的硅晶片（放在晶片架上）

3、带有 8 个弹簧销的有机玻璃板，上有 8 根带弹簧的探针

4、一个欧姆电阻

5、彩色贴纸

重要注意事项

- 所提供的硅晶片，如果掉落或者弯曲，很容易破碎。千万不要触摸或划伤闪亮的金属表面。

实验指南

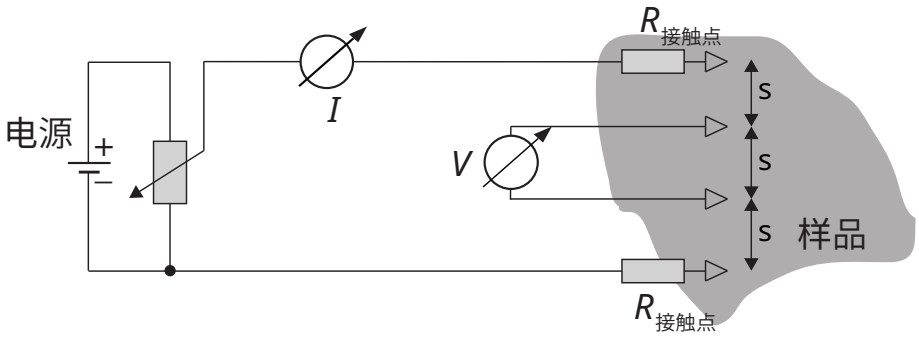
- 在此实验中，信号发生器将被作为一个直流电压源使用。在这种模式下，信号发生器在 voltage（5）和 GDN（7）两端之间输出一个恒定电压。（voltage 意思为“电压”，GDN 的意思为“地”）。括弧中的数字指“实验总指南”中照片中的编号。
- 可以使用螺丝刀，调节左侧标有”adjust voltage（调节电压）（3）“的可调电阻，来调整电压的大小（范围：0 - 5 V）。
- 在进行这部分实验时，通过 toggle switche(拨动开关)（8），确保将信号发生器的扬声器驱动部分关掉。是否关掉，可以通过测量扬声器的 speaker amplitude(振幅监测)（6）和 GND（7）之间的电压来进行检验。如果扬声器驱动部分被关掉的话，则在上述两端之间的电压就是 0。

A 部分：四探针法 (4PP) 的测量 (1.2 分)

要精确测量样品的电阻率，用于测量电压和测量电流的接触点一定要分开。

我们把这种测量技术叫做四探针测量技术（4PP），如图所示。为了尽可能简单，四个接触点按照几何对称来安排，例如电流 I 从一个外接触点（叫做源）流入样品，可能通过各种路径流经样品，然后再从另外一个接触点（叫做漏）流出。在上述两接触点之间取一长度为 s 的路径，测量该段路径上的电压 V。

如果我们搭建一个对称的装置，测量就变得非常简单了。如下图所示：所有的接触点之间的距离都等于 s，样品中央的接触点如下图所示。



I—V 曲线代表了样品的 I—V 特性，通过它可以确定出样品某部分的电阻值。在接下来的实验中我们仅利用 4PP 技术进行测量。开始，我们用线性等间距方法从如图所示的八个探针（接触点）中选取出四个。

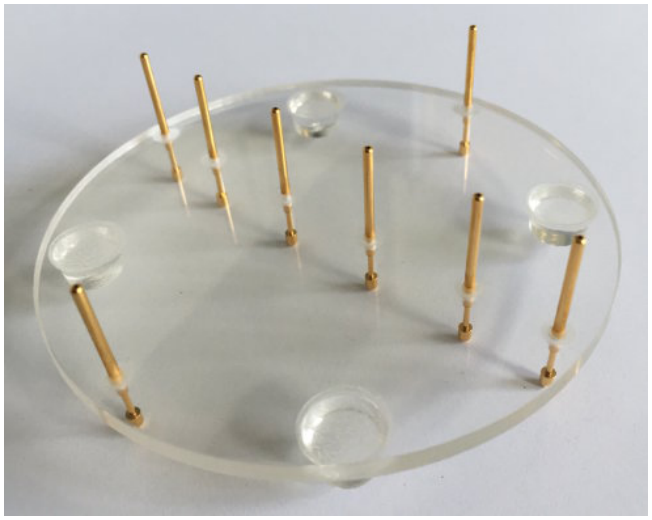


图 2：用于 4PP 测量法的材质为丙烯酸玻璃的测量板，有 4 个塑料脚和 8 个探针或接触点。

下面的测量中，需要用到整张导电纸。

关于下面所有测量的重要提示

- 导电纸的长边是参照边。四个探针排列成一条线，应该与这条参照边平行。

Experiment
China (China)

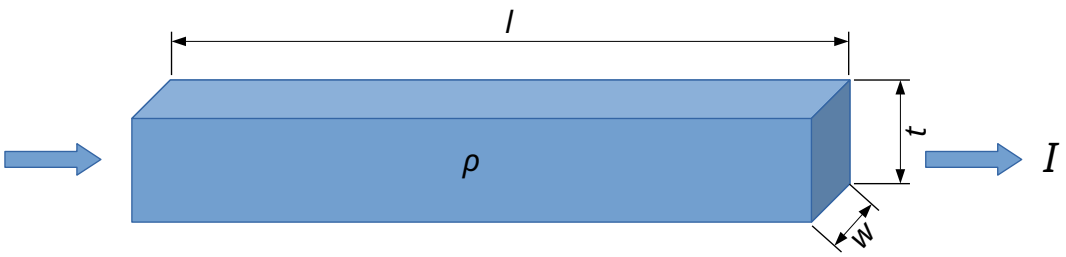
Q1-4

- 小心使用有涂层的一面（黑色的），而不是导电纸的褐色背面！你可以用彩色贴纸标出正确的方向。
- 仔细检查，确信导电纸上没有小洞和划痕。
- 在以下测量中，尽可能地将接触点放置在样品的中心。
- 用一定的力量按压有机玻璃板，以确保接触点与样品有良好的接触，4 个圆形塑料角正好和表面接触。

A.1	四点探针测量法（4PP）：参照图 1 测量 s 长度上的电压降 V，与系统电流 I 间的函数关系，至少共测四组，作一个表，填入数据，在图 A.1 中画出电流 I 随压降 V 变化的关系线。	0.6pt
A.2	由图 A.1 确定整张导电纸的有效电阻 R 的值（$R=\frac{V}{I}$）。	0.2pt
A.3	利用图 A.1 确定 4PP 测量法确定的电阻 R 的不确定度 ΔR。	0.4pt

B 部分 薄片电阻率 (0.3 分)

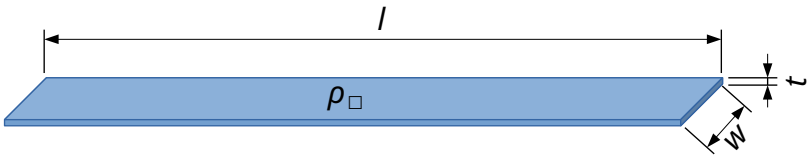
电导率 ρ 是材料的一个重要属性，可以通过给定尺寸和几何结构的三维（3D）导体的电阻值计算得到。这里我们考虑长为 l, 宽为 w，厚度为 t 的条状样品:



上面测量的厚导体的电阻 R 公式如下：

$$R = R_{3D} = \rho \cdot \frac{l}{w \cdot t} \tag{1}$$

依据基本的原理，我们可以定义厚度 $t \ll w$ 和 $t \ll l$ 的二维（2D）导体的电阻：



$$R = R_{2D} = \rho_{\square} \cdot \frac{l}{w}, \tag{2}$$

定义公式（2）时，利用了薄片电阻率关系式 $\rho_{\square} \equiv \rho/t$(读作” rho box”)，其单位为欧姆：$[\rho_{\square}] = 1\,\Omega$

特别重要：公式（2）只有在导体横截面上电流密度均匀并且横截面等势时才成立。在我们这种点接触的情况中是不成立的。可以证明，点接触情况下，取而代之，薄片电阻率与电阻满足下列关系

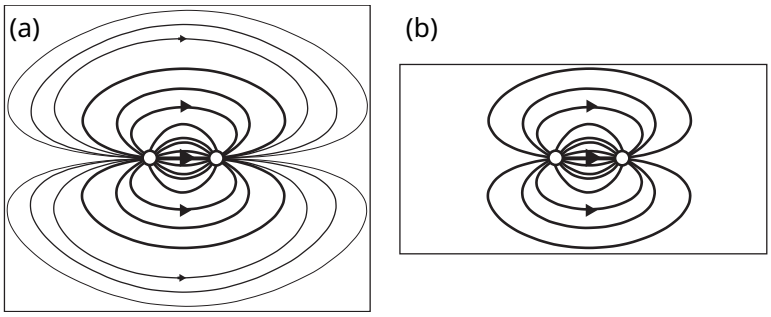
$$\rho_{\square} = \frac{\pi}{\ln(2)} \cdot R \tag{3}$$

上式要求的条件为：$l, w \gg t$.

B.1	计算 A 部分中利用 4PP 方法测量使用的导电纸的薄片电阻率 $\rho_{\square}$。我们命名其为特征值 ρ_{∞}（对应于 A 部分中电阻测量结果 R_{∞}），因为整张样品导电纸的维度对比于接触点间距 s 是相当大的，也就是说:$l, w \gg s$	0.3pt

C 部分 测量样品的不同维度 (3.2 分)

现在还没有考虑样品的维度 w 和 l 是有限的。如果样品变得很小，电压保持恒定时，承载的电流很小：如果我们在两个接触点（白圈）上加上电压，电流将由样品上的所有可能的非交叉的路径流过，如下图（a）所示。用线条表示出这些路径：线条越长则电流越小，电流大小由线条粗细表示。对于小样品，下图（b），加上同样的电压，由于可能的电流路径变少了，总电流将减小。因此测量到的电阻值将增加：



（薄片）电阻率不随着样品的尺寸而改变。因此，为了利用公式（3）将测量的电阻值转化成电阻率，我们需要引入一个修正系数 $f(w/s)$

$$\rho_{\square} = \frac{\pi}{\ln(2)} \cdot \frac{R(w/s)}{f(w/s)}. \tag{4}$$

对于一个长度 $l \gg s$ 的样品来说，系数 f 仅仅依赖于比值 w/s，并且数值大于 1：$f(w/s) \geq 1$。为了简化，我们只关注与宽度 w 的关系，认为我们在实验中的样品长度足够长。我们假定大尺寸情况下，$\rho_{\square}$ 接近正确值，即：

$$R(w/s) = R_{\infty} \cdot f(w/s) \quad \text{with} \quad f(w/s \to \infty) \to 1.0. \tag{5}$$

C.1	用 4PP 方法，对范围在 0.3 到 5.0 之间的四个不同 w/s 值，测量所对应的电阻值 $R(w,s)$，将结果记录在表 C.1 中。请确保样品的长度大于探针间距的五倍：$l > 5s$ ，并且样品的长度 l 总是选取导电纸的同一条长边。 对每一个 w/s 比值，在 4 个不同的电流值下分别测量电压，计算四次测量下的电阻的平均值 $R(w/s)$。 将结果填入表 C.1 中。	3.0pt

C.2	计算这些测量的每一个 $f(w/s)$	0.2pt

D 部分 校正系数：比例法则 (1.9 分)

在 C 部分可看到，在不同的宽度与探头距离之比 w/s 的情况下，测得了的电阻率。从 C 部分获取的数据开始，我们选择以下通用函数来描述测量范围内的数据

$$\text{Generic fit function:} \quad f(w/s) = 1.0 + a \cdot \left(\frac{w}{s}\right)^{b} \tag{6}$$

注意，对于很大的 w/s, $f(w/s)$ 必须是 1.0. 公式（6）中 Generic fit function 的意思为通用拟合函数

D.1	为了用公式 6 和在 C 部分 C 中的数据 $f(w/s)$ 来拟合模型曲线。在线性图 D.1a，半对数图 D.1b，或双对数图 D1.c 中，选择最合适的坐标纸，在图上标出数据点并作图。	1.0pt

D.2	从你做出的拟合线中推算出适合的参数 a 和 b。	0.9pt

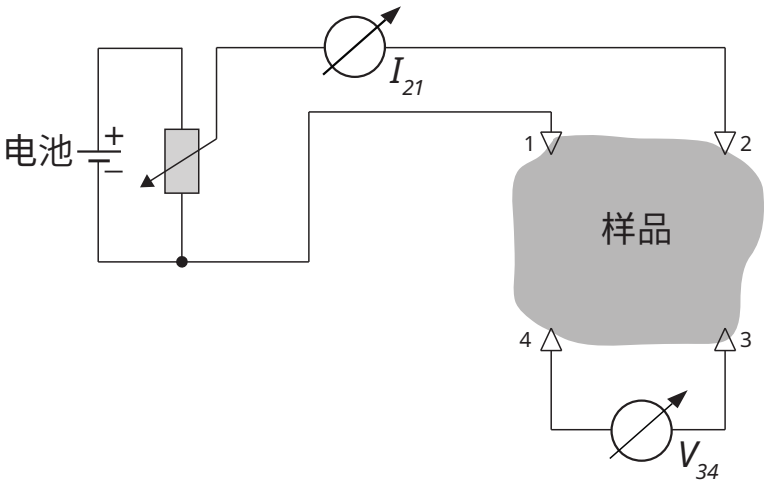
E 部分：硅晶片和范德堡法 (3.4 分)

在半导体工业中，半导体与薄金属层的薄片电阻率大小是很重要的知识，因为它决定了元器件特性。接下来你会测量一个硅晶圆片。这个硅晶圆片的亮面镀了一层很薄的铬金属。

顺着箭头方向旋转（RELEASE 意思为打开），打开硅晶圆片盒取出晶圆片。请注意，硅晶圆片如果掉落或弯曲，就可能破裂。也不要碰触或刮伤闪光的金属薄膜表面。测量硅晶圆片时，请放置于桌上，亮面向上。

E.1	使用跟之前一样的四探针法，测量电压 V 随电流 I 变化的数据，填入表中。 在答题卡上写下你的硅晶圆片盒上的编号，该编号标在塑料晶片架上。	0.4pt
E.2	在图 E.2 上作图，确定电阻 $R_{4\mathrm{PP}}$ 的值。	0.4pt
E.3	为了确定一个圆形样品如硅晶圆片的电性校正值，我们近似样品的有效宽度 w 为硅晶圆片的直径 D=100 mm。在这个假设下计算比例值 w/s。使用拟合函数（6）与参数 a 与 b 来决定硅晶圆片电性的校正因子 $f(w/s)$ 的值。	0.2pt
E.4	使用公式（4）计算铬薄膜层的薄片电阻率 $\rho_{\square}$	0.1pt

为了避开使用上述的几何校正因子，精确计算薄片电阻率，飞利浦公司的工程师 L.J. 范德堡发展出一个简单的测量法。将四个探针放在任意形状的样品边缘，如图所示（1 到 4 号为探针编号）。电流在相邻两探针间流动（如 1 与 2 之间）时，测量 3 与 4 之间的电压。这样可以得到电阻值 $R_{I,V} = R_{21,34}$



由于对称的原因 $R_{21,34} = R_{34,21}$ 和 $R_{14,23} = R_{23,14}$，范德堡指出，对于任意而简单的连接形状（无孔）的样品并采用点状接触，以下方程成立：

$$e^{-\pi R_{21,34}/\rho_{\square}} + e^{-\pi R_{14,23}/\rho_{\square}} \equiv 1. \tag{7}$$

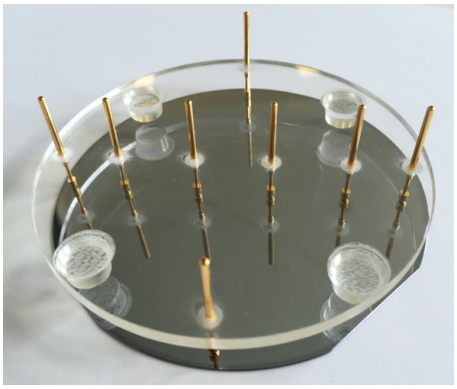


图 3：镀有金属膜层的硅圆晶片上的 4PP 元件。注意在圆晶片的右手边的切口。这种切口被称为对位切边。

如图所示连接四个构成正方形的弹簧探针。将两个相邻的弹簧探针连接到有电流表的电流源，剩下的两个弹簧探针与电压表连接。旋转这个正方形直到其一个边平行于晶片的对位切边。

E.5	绘制流过触点的电流的方向，和晶片的对位切边的方向。测量至少 6 个不同电流 I 值下的电压值 V，要求电流 I 大致等间隔。将结果填入表 E.5。	0.6pt
E.6	使电流垂直于上一步中的电流方向，重复上述实验过程，将测量结果填入表 E.6。	0.6pt
E.7	用不同的颜色和（或）符号，将所有数据在一张图 E.7 上绘图。确定两条曲线的平均值 $\langle R \rangle$。	0.5pt
E.8	使用 $\langle R \rangle$ 替换所有电阻 $R_{kl,mn}$，解关于 $\rho_{\square}$ 的方程（7），并计算铬薄膜层的薄片电阻率 $\rho_{\square}$(vdP)	0.4pt
E.9	比较线性法 (E.**4**) 和范德堡法（E.8）两种不同测量方法的结果。以百分比的相对误差方式，给出两种方法的差别。	0.1pt
E.10	在铬（Cr）薄膜层具有 8nm 的标称厚度。使用这个值和范德堡方法的最终结果，通过方程（1）和（2），来计算铬的电阻率。	0.1pt

跳动的珠子—相变和不稳定性的模型（10 分）

在开始进行本实验之前，请认真阅读在另外一个单独信封里装着的“实验考试总指南”。

引言

在日常生活中，相变经常可见。比如，水具有固态、液态和气态。这些不同的态可以通过相变来区分，不同状态下，材料中分子的集体行为将会改变。相变总是与转变温度相关，在转变温度时产生相变，比如，水的结冰温度和汽化温度。

其实，相变在很多系统中很广泛地发生着，如磁体在转变温度下，宏观状态从顺磁铁转化成铁磁态；又比如，在超导体在转变温度下，从常规导体转化为超导态。

所有这些相变现象，在通常情况下，都可以通过引入一个所谓的有序参数来进行描述。例如，在磁铁中，有序参数与原子磁矩的排列相关，而磁矩的排列决定了宏观的磁性。

在所谓的连续相变过程中，在临界温度以上，有序参数总是等于 0。在临界温度以下，随着温度的降低，有序参数逐渐连续的增大，图 1 给出了磁铁的有序参数随温度变化的情况。图中也给出了磁铁的微观有序度和无序度：在铁磁态下，所有磁矩是一顺排列的，这导致了宏观的磁性；而在顺磁态下，磁矩是随意取向的，导致了其宏观磁性为零。

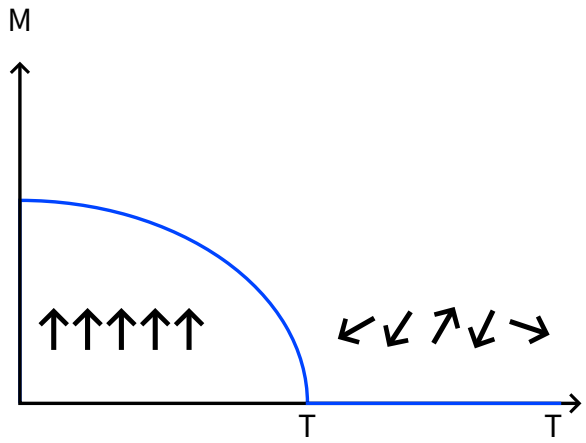


图 1，在相变过程中，有序参数 M 随着温度的变化。在临界温度 Tcrit 以下，有序参数不再为 0，而是随着温度降低而增加；而在临界温度 Tcrit 以上，有序参数为 0。

在连续相变中，靠近相变时的有序参数满足一个幂函数关系，如：在临界温度 T_{crit} 以下，磁铁的磁化强度 M（有序参数）满足下式：

$$M \begin{cases} \sim (T_{\mathrm{crit}} - T)^b, & T < T_{\mathrm{crit}} \\ = 0, & T > T_{\mathrm{crit}} \end{cases} \tag{1}$$

式中，T 为温度。更令人惊讶的是，这个幂函数规律是一个普遍规律，幂函数公式在很多相变中都适用。

实验任务

我们通过一个易于观察的简单例子来研究连续相变的一些特性，例如不稳定性是如何影响微观粒子（珠子）的集体行为的，进而是如何影响相变的，以及宏观变化是如何依赖于微观粒子的激发的。

在通常的相变中，相变是由温度激发的。在我们的例子中，这种激发来自于扬声器提供给粒子加速振动的动能。对应于相变的宏观变化现象是我们将要研究的内容，实验中是将珠子放进圆柱体中的两分隔部分之一，圆柱体中有一个小隔板将其平分成两部分。

珠子目前分布在柱体中的两分隔部分之一，增加扬声器的振动幅度，你会发现，最终珠子均匀地分布在两个分隔部分中。这表明，已经加热到临界温度以上。

你的实验目的就是：确定相变模型中的临界指数。

材料列表

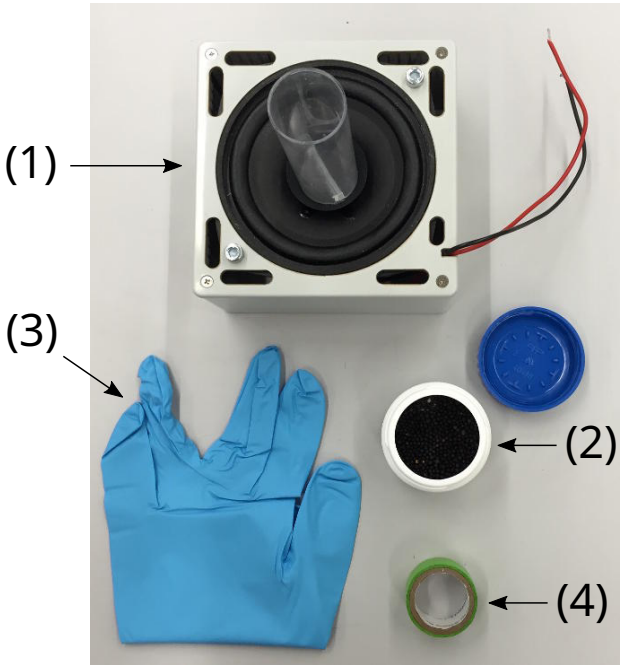


图 2 本实验还需使用的元件

1. 扬声器，其顶端装有一个圆柱状塑料管
2. 大约 100 粒珠子（装在塑料盒子中）
3. 手套
4. 胶带

重要预防措施

- 对安装在扬声器上的塑料管，请勿施加多余的侧压。注意，扬声器上的膜撕裂或者扯下塑料管，将没有替代品可提供

- 不使用时，关闭扬声器的电源，以免电池产生不必要消耗。
- 在本实验中，一个 4Hz 锯齿状信号输出到扬声器引线，该引线连接到信号发生器侧面。
- 锯齿信号的幅度可以通过右边的可变电阻器进行调整，可变电阻器标记有扬声器振幅【speaker amplitude (4)】。一个正比于信号振幅的直流电压输出到扬声器的振幅监控插口（6）（相对于接地插口 GND（7）。括弧中的数字指的是实验考试总指南中的照片。
- 扬声器上的膜是很精细的，确保其上没有不需要的压力，包括垂直的和侧向的压力。

A 部分- 临界激发振幅（3.3 分）

在开始做本题实验前，将扬声器引线连接到信号发生器上的侧面（保证连接的极性正确）。把一些（例如 50 粒）珠子放在固定在扬声器上的圆柱塑料管内，并用从手套切下的一块塑料膜封闭圆柱管顶部，保证珠子保持在圆柱管内。打开开关，用提供的螺丝起子转动右边标记为“扬声器振幅”（4）的可变电阻器，以调整振幅。在不同的振幅，观察珠子的分布情况。

第一个任务是确定出相变的临界激发振幅。为此，你必须确定两分隔部分的珠子数目 N_1 和 N_2（设 $N_1 \leq N_2$），它们为振幅的 A_D 函数，即是在扬声器振幅插口 (6) 测量到的电压。这个电压正比于驱动扬声器的锯齿信号的振幅。

为了使你研究的珠子总处在运动状态，只研究对应于扬声器振幅电压超过 0.7V 情况下的振幅。

- 开始时不用对珠子计数，缓慢改变电压，观察实验现象。由于静电原因，可能会有一些珠子粘在塑料管的底部。

A.1	改变振幅 A_D，多次测量容器两分隔部分的珠子数 N_1 和 N_2，记录在表 A.1 中。	1.2pt
A.2	计算多次测量值 N_1 和 N_2 的标准偏差，并将结果写在**表 A.1 中**。在**图 A.2 中**，画出 N_1 和 N_2 与 A_D（作为所显示的振幅）的函数关系图，图中要包括所有数据点的不确定度。	1.1pt
A.3	根据你作出的图，确定临界显示振幅 $A_{D,\mathrm{crit}}$，此时 $N_1 = N_2$，达到稳定的状态。	1pt

B 部分-校准（3.2 分）

显示振幅 A_D 对应施加于在扬声器上的电压。然而，扬声器振动的最大移动距离 A 才是我们真正感兴趣的物理量，因为它和珠子的激发有多强有关。因此，你必须对显示振幅与对应的激发振幅进行校定。出于这个目的，你可以所提供的任何材料及工具。

B.1	画出你用来测量激发振幅的实验装置示意图，这里激发振幅为一个周期内扬声器的最大移动距离 A（以 mm 表示）	0.5pt
B.2	测出激发振幅 A（以 mm 表示），保留适当的小数位，在**表 B.2** 记录下激发振幅 A 和显示振幅 A_D 的值，最后标示出不确定度。	0.8pt
B.3	将你的数据画在图 B.3 中，包括所有数据的不确定度。	1.0pt
B.4	利用作出的曲线，通过适当的拟合确定校定 $A(A_D)$ 函数关系中的参数。	0.8pt
B.5	确定珠子的临界激发振幅 A_{crit}。	0.1pt

在我们的系统中，温度与输入的激发动能相对应。这个能量正比于扬声器速度的平方，即，$v^2 = A^2 f^2$，其中 f

是振荡频率。现在，我们将验证这种依赖关系，并且确定有序参数行为的幂函数关系式中的幂指数 b。（见公式 1）

C.1	非平衡量 $\left\|\frac{N_1-N_2}{N_1+N_2}\right\|$ 是描述我们系统的一个很好的有序参数。因为在临界振幅以上，其值为 0，在低激发振幅下其值等于 1 。计算这个有序参数，它是激发振幅 A 的函数。将各个物理量的结果填入表 **C.1**.	1.1pt
C.2	在图 **C.2 中，用**双对数坐标，画出非平衡量 $\left\|\frac{N_1-N_2}{N_1+N_2}\right\|$ 随 $\left\|A_{\mathrm{crit}}^2-A^2\right\|$ 参数变化的关系线。使用表 C.1 中的数据进行计算。图中的数据似乎不满足线性关系，但是可以通过线性回归，以便与临界指数公式相匹配。	1pt
C.3	确定出指数 b 并估计误差。	1.4pt

Problem 1 : Solution/marking scheme – Two Problems in Mechanics *(10 points)*

Part A. The Hidden Disk (3.5 points)

A1 (0.8 pt) Find an expression for b as a function of the quantities (1), the angle ϕ and the tilting angle Θ of the base.

Solution A1: **[0.8]**

Geometric solution: use that torque with respect to point of contact is 0 $\Rightarrow$ center of gravity has to be vertically above point of contact.

$$\sin\phi = \frac{D}{b} \qquad \textbf{0.3}$$

$$\sin\Theta = \frac{D}{r_1} \qquad \textbf{0.3}$$

Here D may be called another name. Solve this:

$$\sin\phi = \frac{r_1}{b}\sin\Theta \;\Rightarrow\; b = \frac{r_1\sin\Theta}{\sin\phi} \qquad \textbf{0.2}$$

Alternative: Torque and forces with respect to another point: **[0.8]**
Correct equation for torque **0.3**
Correct equation for force **0.3**
Correct solution **0.2**

A2 (0.5 pt) Find the equation of motion for φ. Express the moment of inertia I_S of the cylinder around its symmetry axis S in terms of T, b and the known quantities (1). You may assume that we are only disturbing the equilibrium position by a small amount so that φ is always very small.

Solution A2: **[0.5]**

Write some equation of the form $\ddot{\varphi} = -\omega^2\varphi$ **0.1**

Writing an equation of the form $\varphi = A\cos\omega t$ is also correct.
Two solutions:

1. Kinetic energy: $\frac{1}{2}I_S\dot{\varphi}^2$ and potential energy: $-bMg\cos\varphi$. Total energy is conserved, and differentiation w.r.t. time gives the equation of motion.
2. Angular equation of motion from torque, $\tau = I_S\ddot{\varphi} = -Mgb\sin\varphi$.

Correct equation (either energy conservation or torque equation of motion) **0.3**
Final answer

$$T = 2\pi\sqrt{\frac{I_S}{Mgb}} \;\Rightarrow\; I_S = \frac{MgbT^2}{4\pi^2} \qquad \textbf{0.1}$$

(Derivation:

$$\Rightarrow \ddot{\varphi} = -\frac{bMg}{I_S}\sin\varphi \simeq -\frac{bgM}{I_S}\varphi$$

so that

$$\omega^2 = \frac{bgM}{I_S}$$

)

A3 (0.4 pt) Find an expression for the distance d as a function of b and the quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.

Solution A3: **[0.4]**

Some version of the center of mass equation, e.g.

$$b = \frac{dM_2}{M_1 + M_2}$$ **0.2**

correct solution:

$$d = \frac{bM}{\pi h_2 r_2^2(\rho_2 - \rho_1)}$$ **0.2**

A4 (0.7 pt) Find an expression for the moment of inertia I_S in terms of b and the known quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.

Solution A4: **[0.7]**

correct answer for moment of inertia of homogeneous disk

$$I_1 = \frac{1}{2}\pi h_1 \rho_1 r_1^4$$ **0.2**

Mass wrong **-0.1**
Factor 1/2 wrong in formula for moment of inertia of a disk **-0.1**
Correct answer for moment of inertia of 'excess' disk:

$$I_2 = \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4$$ **0.2**

Using Steiner's theorem:

$$I_S = I_1 + I_2 + d^2\pi r_2^2 h_2(\rho_2 - \rho_1)$$ **0.1**

correct solution:

$$I_S = \frac{1}{2}\pi h_1\rho_1 r_1^4 + \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4 + \frac{b^2M^2}{\pi r_2^2 h_2(\rho_2 - \rho_1)}$$ **0.2**

In terms of d rather than b gives 0.1pts rather than 0.2pts for the final answer: **0.1**

$$I_S = \frac{1}{2}\pi h_1 \rho_1 r_1^4 + \frac{1}{2}\pi h_2(\rho_2 - \rho_1) r_2^4 + d^2 \pi r_2^2 h_2(\rho_2 - \rho_1)$$

A5 (1.1 pt) Using all the above results, write down an expression for h_2 and r_2 in terms of b, T and the quantities (1). You may express h_2 as a function of r_2.

Solution A5: **[1.1]**

It is not clear how exactly students will attempt to solve this system of equations. It is likely that they will use the following equation:

$$M = \pi r_1^2 h_1 \rho_1 + \pi r_2^2 h_2(\rho_2 - \rho_1) \ . \qquad \textbf{0.3}$$

solve I_S for r_2^2:

$$r_2^2 = \frac{2}{M - \pi r_1^2 h_1 \rho_1}\left(I_S - \frac{1}{2}\pi h_1 \rho_1 r_1^4 - b^2 \frac{M^2}{M - \pi r_1^2 h_1 \rho_1}\right) \qquad \textbf{0.4}$$

replace I_S by T:

$$I_S = \frac{M g b T^2}{4\pi^2} \qquad \textbf{0.1}$$

solve correctly for r_2:

$$r_2 = \sqrt{\frac{2}{M - \pi r_1^2 h_1 \rho_1}\left(M\frac{bgT^2}{4\pi^2} - \frac{1}{2}\pi h_1 \rho_1 r_1^4 - b^2 \frac{M^2}{M - \pi r_1^2 h_1 \rho_1}\right)} \qquad \textbf{0.1}$$

write down an equation for h_2 along the lines of $M = \pi r_1^2 \rho_1 h_1 + \pi r_2^2(\rho_2 - \rho_1)h_2$ and solve it correctly:

$$h_2 = \frac{M - \pi r_1^2 \rho_1 h_1}{\pi r_2^2(\rho_2 - \rho_1)} \qquad \textbf{0.2}$$

Part B. Rotating Space Station (6.5 points)

B1 (0.5 pt) At what angular frequency ω_{ss} does the space station rotate so that the astronauts experience the same gravity g_E as on the Earth's surface?

Solution B1: **[0.5]**

An equation for the centrifugal force along the lines of

$$F_{ce} = m\omega^2 r \qquad \textbf{0.1}$$

Balancing the forces, correct equation

$$g_E = \omega_{ss}^2 R$$ **0.2**

Correct solution

$$\omega_{ss} = \sqrt{g_E/R}$$ **0.2**

B2 (0.2 pt) Assuming that on Earth gravity is constant with acceleration g_E, what would be the angular oscillation frequency ω_E that a person on Earth would measure?

Solution B2: **[0.2]**

Realize that result is independent of g_E **0.1**

Correct result:

$$\omega_E = \sqrt{k/m}$$ **0.1**

B3 (0.6 pt) What angular oscillation frequency ω does Alice measure on the space station?

Solution B3: **[0.6]**

some version of the correct equation for force

$$F = -kx \pm m\omega_{ss}^2 x$$ **0.2**

getting the sign right

$$F = -kx + m\omega_{ss}^2 x$$ **0.2**

Find correct differential equation

$$m\ddot{x} + (k - m\omega_{ss}^2)x = 0$$ **0.1**

Derive correct result

$$\omega = \sqrt{k/m - \omega_{ss}^2}$$ **0.1**

Using g_E/R instead of ω_{ss}^2 is also correct.

B4 (0.8 pt) Derive an expression of the gravity $g_E(h)$ for small heights h above the surface of the Earth and compute the oscillation frequency $\tilde{\omega}_E$ (linear approximation is enough). The radius of the Earth is given by R_E.

Solution B4: **[0.8]**

$$g_E(h) = -GM/(R_E + h)^2$$ **0.1**

linear approximation of gravity:

$$g_E(h) = -\frac{GM}{R_E^2} + 2h\frac{GM}{R_E^3} + \dots$$ **0.2**

Realize that $g_E = GM/R_E^2$:

$$g_E(h) = -g_E + 2hg_E/R_E + \dots$$ **0.1**

Opposite sign is also correct, as long as it is opposite in both terms.
Realize what this means for force, i.e. that the constant term can be eliminated by shifting the equilibrium point:

$$F = -kx + 2xmg_E/R_E$$ **0.2**

Find correct differential equation

$$m\ddot{x} + (k - 2mg_E/R_E)x = 0$$ **0.1**

correct result

$$\tilde{\omega}_E = \sqrt{k/m - 2g_E/R_E}$$ **0.1**

No points are deducted if student answers with $\tilde{\omega}_E/(2\pi)$ because "oscillation frequency" might also be interpreted as inverse period.

B5 (0.3 pt) For what radius R of the space station does the oscillation frequency ω match the oscillation frequency $\tilde{\omega}_E$ on the surface of the Earth? Express your answer in terms of R_E.

Solution B5: **[0.3]**

Write down equation

$$\omega_{ss}^2 = 2g_E/R_E$$ **0.1**

Solve

$$R = R_E/2$$ **0.2**

If GM/R_E^2 rather than g_E is used, give only *0.1pt*.

B6 (1.1 pt) Calculate the horizontal velocity v_x and the horizontal displacement d_x (relative to the base of the tower, in the direction perpendicular to the tower) of the mass at the moment it hits the floor. You may assume that the height H of the tower is small, so that the acceleration as measured by the astronauts is constant during the fall. Also, you may assume that $d_x \ll H$.

Solution B6: **[1.1]**

There are several possible solutions.
Solution one – Using Coriolis force

- Velocity v_x

  Equation for Coriolis force with correct velocity:

  $$F_C(t) = 2m\omega_{ss}^2 Rt\omega_{ss} = 2m\omega_{ss}^3 Rt$$ **0.1**

  Integrate this, or realize that it is like uniform acceleration for the velocity:

  $$v_x(t) = \omega_{ss}^3 Rt^2$$ **0.2**

  plug in correct value for

  $$t = \sqrt{2H/\omega_{ss}^2 R}$$ **0.2**

  overall correct result

  $$v_x = 2H\omega_{ss}$$ **0.1**

- The displacement d_x:

  Integrate $v_x(t)$:

  $$d_x = \frac{1}{3}R\omega_{ss}^3 t^3$$ **0.3**

  Instead of integrating, students may simply 'average' by taking $\frac{1}{2}$ of the final velocity. This gives a factor of $\frac{1}{2}$ instead of $\frac{1}{3}$. *Deduct a total of 0.1 pts for this.* **-0.1**

  Plug in value for t

  $$d_x = \frac{1}{3}R\omega_{ss}^3(2H/\omega_{ss}^2 R)^{3/2} = \frac{1}{3}2^{3/2}H^{3/2}R^{-1/2} = \frac{1}{3}\sqrt{\frac{8H^3}{R}}$$ **0.2**

Solution two – Using inertial frame This solution is similar to the way to solve B7, but needs more complicated approximations than Solution one.

- v_x

  Here ϕ denotes the angle swept by the mass and α the angle the astronauts (and tower) has rotated when the mass lands on the floor, see

  Initially the velocity of the mass in an inertial frame is $v_x = \omega_{ss}(R - H)$. **0.1**

  When the mass lands, the x-direction has been rotated by ϕ so the new horizontal velocity component is then

  $$\omega_{ss}(R - H)\cos\phi$$ **0.1**

  (Student may also write $\cos\alpha$ instead of $\cos\phi$, since $d_x \ll H$.)

  $$\cos\phi = \frac{R - H}{R} = 1 - \frac{H}{R}$$ **0.1**

  Transforming to the rotating reference frame, one needs to subtract $\omega_{ss}R$. **0.1**

  Finally in the reference frame of the astronauts

  $$v_x = \omega_{ss}R\left(1 - \frac{H}{R}\right)^2 - \omega_{ss}R \approx \omega_{ss}R\left(1 - 2\frac{H}{R}\right) - \omega_{ss}R = -2\omega_{ss}H$$ **0.2**

  The sign of the velocity depend on the choice of reference direction, so a positive sign is also correct.

- d_x

  With the notation from the calculation of v_x

$$d_x = (\alpha - \phi)R$$ **0.1**

$$\phi = \arccos\left(1 - \frac{H}{R}\right)$$
$$\alpha = \omega_{ss} t$$

  where t is the fall time of the mass, which is given by

$$t = \frac{\sqrt{R^2 - (R-H)^2}}{\omega_{ss}(R-H)}$$ **0.1**

  (see solution to B7)

  Writing $\xi \equiv H/R$ this means

$$d_x = \left[\frac{\sqrt{1-(1-\xi)^2}}{1-\xi} - \arccos(1-\xi)\right] R$$ **0.3**

  which is a valid end answer to the problem. It is possible, but not necessary, to approximate this for small ξ:

$$\arccos(1-\xi) \approx \sqrt{2\xi}\left(1 + \frac{\xi}{12}\right)$$

  which after insertion into the equation for d_x and approximation of small ξ yields the same result as in Solution one:

$$d_x = \frac{2}{3}\sqrt{\frac{2H^3}{R}}$$

  If this end answer misses the factor 2/3, deduct 0.1 points. **-0.1**

Solution three – Inertial frame with geometry trick

This is an alternative solution to obtain d_x

The mass travels the distance l, and during the fall the space station rotates by ϕ, see Figure 2. According to the intersecting chord theorem,

$$l^2 = H(2R - H)$$ **0.1**

The rotated angle is $\phi = \omega_{ss} t$ where

$$t = \frac{l}{R-H}$$ **0.1**

is the fall time. Thus

$$\phi = \frac{\sqrt{H(2R-H)}}{R-H}$$ **0.1**

$$\frac{d}{R} = \phi - \arcsin\frac{l}{R} = \frac{\sqrt{H(2R-H)}}{R-H} - \arcsin\sqrt{x(2-x)}$$ **0.1**

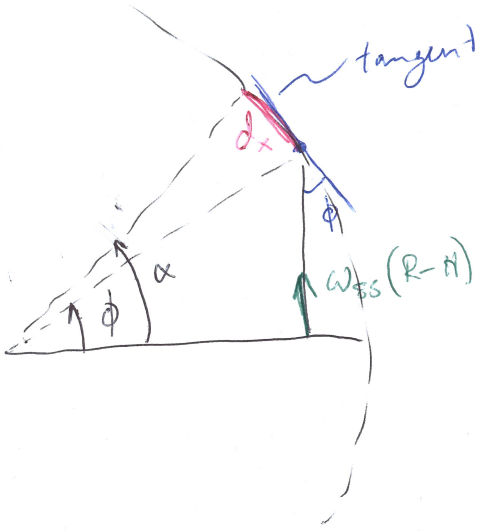


Figure 1: Notation for solution two

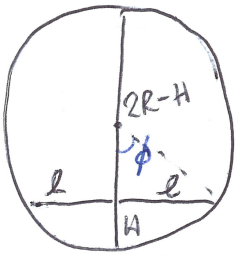


Figure 2: Notation for solution three.

Denote $x \equiv H/R$ and $y \equiv \sqrt{x(2-x)}$. Since

$$\arcsin y \approx y + \frac{y^3}{6}$$

one gets

$$\frac{d}{R} \approx y(1+x) - y - y^3/6 = y(x - y^2/6) \approx 2xy/3 \approx 2x\sqrt{2x}/3 = \frac{2}{3}\sqrt{\frac{2H^3}{R}}$$

Final answer **0.1**

B7 (1.3 pt) Find a lower bound for the height of the tower for which it can happen that $d_x = 0$.

Solution B7: **[1.3]**

The key is to use a non-rotating frame of reference. If the mass is released close enough

to the center, its linear velocity will be small enough for the space station to rotate more than 2π before it hits the ground.
The velocity is given by

$$v = \omega_{ss}(R - H) \quad \textbf{0.1}$$

distance d that the mass flies before hitting the space station

$$d^2 = R^2 - (R - H)^2 \quad \textbf{0.1}$$

use non-rotating frame of reference to obtain time t until impact

$$t = d/v = \frac{\sqrt{R^2 - (R - H)^2}}{\omega_{ss}(R - H)} \quad \textbf{0.1}$$

Now there are several possible ways to relate H and the rotated angle ϕ of the space station:

Solution one

$$t = \frac{R \sin\phi}{\omega_{ss} R \cos\phi} \quad \textbf{0.2}$$

This time must match $t = \phi/\omega_{ss}$. Obtain the equation

$$\phi = \tan\phi \quad \textbf{0.2}$$

Realizing that there is an infinite number of solutions. **0.2**
This equation has one trivial solution $\phi = 0$, next solution is slightly less than $3\pi/2$ which corresponds to the case $H > R$ (and is thus not correct). The one that gives a lower bound for H is the third solution

$$\phi \approx 5\pi/2$$

The equation $\phi = \tan\phi$ can be solved graphically or numerically to obtain a close value ($\phi = 7.725$ rad) which means

$$H/R = (1 - \cos\phi) \approx 0.871$$

Give points if the method is correct, depending on the value of H/R found, according to these intervals: **0.4**
$0.85 \leq H/R \leq 0.88$: *0.4 pts*
$0.5 \leq H/R < 0.85$: *0.3 pts*
$0 < H/R < 0.5$ or $H > 0.88$: *0.2 pts*
$H = 0$ or method is incorrect: *0 pts*

Solution two
relation between H and rotated angle ϕ

$$\frac{R - H}{R} = \cos\phi \quad \textbf{0.2}$$

obtain equation of the form

$$\frac{H}{R} = 1 - \cos\left(\frac{\sqrt{1 - (1 - H/R)^2}}{1 - H/R}\right) \quad \textbf{0.2}$$

Figure 3 gives a plot of $f(x) = 1 - \cos\left(\frac{\sqrt{1-(1-x)^2}}{1-x}\right)$. The goal is to find an approximate solution for the second intersection. The first intersection is discarded – it is introduced because of $\cos\phi = \cos(-\phi)$ and corresponds to a situation with $H > R$.
Realizing that there is an infinite number of solutions. **0.2**

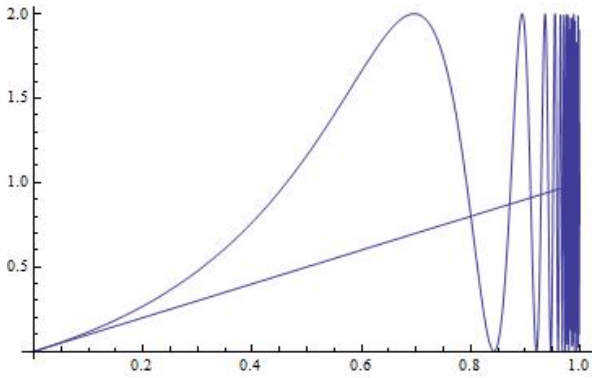


Figure 3: Plot of $f(H/R)$ and H/R

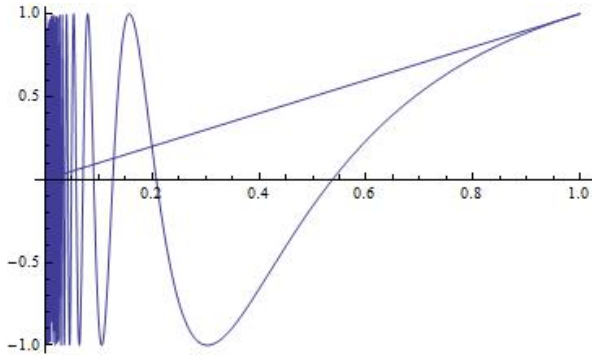


Figure 4: Plot of $g(x)$ and x

- introduce new variable $x := 1 - H/R$, so that the equation becomes

$$x = \cos(\sqrt{1-x^2}/x) =: g(x)$$

- $g(x)$ is then smaller than x up to the first solution. In particular it is negative in some region (see figure 4). Finding the third zero thus gives a lower bound for the solution:

$$\frac{\sqrt{1-x^2}}{x} = 5\pi/2$$

- give lower bound

$$x = 1/\sqrt{25\pi^2/4+1} \Rightarrow H = R(1 - 1/\sqrt{25\pi^2/4+1}) \approx 0.874$$

Note: the actual result is $H/R = 0.871\ldots$.

Use the same points for the numerical answer as was mentioned in solution one. **0.4**

If the student plots f rather than g, find solution to $f = 1$: is equivalent to the solution above. *Give same number of points.*

It is also possible to use $\cos\left(\frac{\sqrt{1-x^2}}{x}\right) = \sin(1/x)$.

B8 (1.7 pt) Alice pulls the mass a distance d downwards from the equilibrium point $x = 0$, $y = 0$, and then lets it go (see figure 4).

- Give an algebraic expression of $x(t)$ and $y(t)$. You may assume that $\omega_{ss}d$ is small.
- Sketch the trajectory $(x(t), y(t))$, marking all important features such as amplitude.

Solution B8: **[1.7]**

Note: we did not specify the overall sign of the Coriolis force. Give same amount of points if using opposite convention, but it has to be consistent! Otherwise: subtract *0.1pt* for each instance of inconsistency. **-0.1**

Students are allowed to express everything in terms of ω, they don't need to write $\sqrt{k/m - \omega_{ss}^2}$ explicitly. Deduct *0.1pt* however if they use k/m instead of ω.. **-0.1**

Realize that $y(t)$ is standard harmonic oscillation:

$$y(t) = A\cos\omega t + B$$ **0.1**

Give correct constants from initial conditions

$$y(t) = -d\cos\omega t$$ **0.2**

Correct expression for $v_y(t)$:

$$v_y(t) = -d\omega\sin\omega t$$ **0.1**

Coriolis force in x-direction

$$F_x(t) = 2m\omega_{ss}v_y(t) = -2m\omega_{ss}d\omega\sin\omega t$$ **0.2**

Realize that this implies that $x(t)$ is also a harmonic oscillation... **0.1**

...but with a constant movement term superimposed: vt **0.1**

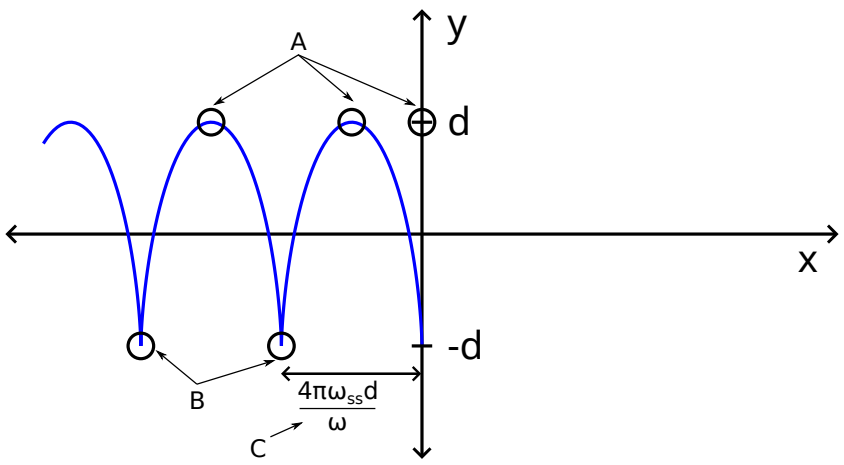
getting the correct amplitude:

$$A = \frac{2\omega_{ss}d}{\omega}$$ **0.1**

Correct answer with correct initial conditions:

$$x(t) = \frac{2\omega_{ss}d}{\omega}\sin\omega t - 2\omega_{ss}dt$$ **0.2**

Sketch:



Correct qualitative sketch:	
periodic motion	**0.1**
overall constant movement	**0.1**
B): cusps	**0.1**
And additionally correct quantitative sketch:	
A)+B): peaks and cusps are at $y = \pm d$	**0.1**
C): cusps are at distance $\Delta x = \frac{4\pi\omega_{ss}d}{\omega}$ from each other	**0.2**

Problem 2 : Solution/marking scheme – Nonlinear Dynamics in Electric Circuits *(10 points)*

Part A. Stationary states and instabilities (3 points)

Solution A1: **[0.4]**

By looking at the $I - V$ graph, we obtain

$$R_{\text{off}} = 10.0\,\Omega, \quad \textbf{0.1}$$

$$R_{\text{on}} = 1.00\,\Omega, \quad \textbf{0.1}$$

$$R_{\text{int}} = 2.00\,\Omega, \quad \textbf{0.1}$$

$$I_0 = 6.00\,A. \quad \textbf{0.1}$$

Note: No penalty for the number of digits in this question

Solution A2: **[1]**

Kirchoff law for the circuit (U is the voltage of the bistable element):

$$\mathcal{E} = IR + U \quad \textbf{0.1}$$

This yields

$$I = \frac{\mathcal{E} - U}{R} \quad \textbf{0.1}$$

Hence, stationary states of the circuit are intersections of the line defined by this equation and the $I - V$ graph of X. **0.2**

For $R = 3.00\,\Omega$, one always gets exactly one intersection. **0.2**
For $R = 1.00\,\Omega$, one gets 1, 2 or 3 intersections depending on the value of $\mathcal{E}$. **0.4**
The following table summarizes the number of points granted for possible answers to the last subquestion with $R = 1.00\,\Omega$:

Possible answer	1	2	3	1,3	1,2	2,3	1,2,3
Points	0	0	0.2	0.3	0	0.2	0.4

Solution A3: **[0.6]**

The stationary state is on the intermediate branch, one can thus use the corresponding equation: **0.2**

$$I_{\text{stationary}} = \frac{\mathcal{E} - R_{\text{int}} I_0}{R - R_{\text{int}}} \quad \textbf{0.1}$$
$$= 3.00\,A \quad \textbf{0.1}$$

$$U_{\text{stationary}} = R_{\text{int}}(I_0 - I) \quad \textbf{0.1}$$
$$= 6.00\,V \quad \textbf{0.1}$$

Extra (non-physical) stationary states on the switched on and/or switched off branches lead to a penalty of 0.2 point.

Solution A4: **[1]**

Any correct modeling such as the following: **0.5**
The Kirchoff law for the circuit reads

$$\mathcal{E} = IR + U_X + L\frac{dI}{dt} = IR + (I_0 - I)R_{\text{int}} + L\frac{dI}{dt}$$

This implies

$$L\frac{dI}{dt} = \mathcal{E} - I_0 R_{\text{int}} - (R - R_{\text{int}})I$$

The separation between two cases is of importance, especially because of the relative sign of dI/dt:
If $I > I_{\text{stationary}}$, we have $dI/dt < 0$ and I decreases. **0.2**
If $I < I_{\text{stationary}}$, we have $dI/dt > 0$ and I increases. **0.2**
Note: Formulas with time derivatives are not essential, any other correct justification is accepted.
We conclude that the stationary state is stable. **0.1**

Note: The checkbox gives 0.1 points if "stable" is checked, regardless of the previous reasoning (also if there is nothing). A wrong reasoning leading to check the "unstable" option doesn't however give any point for the checkbox.

Part B. Bistable non-linear elements in physics and engineering: radio transmitter (5 points)

Solution B1: **[1.8]**

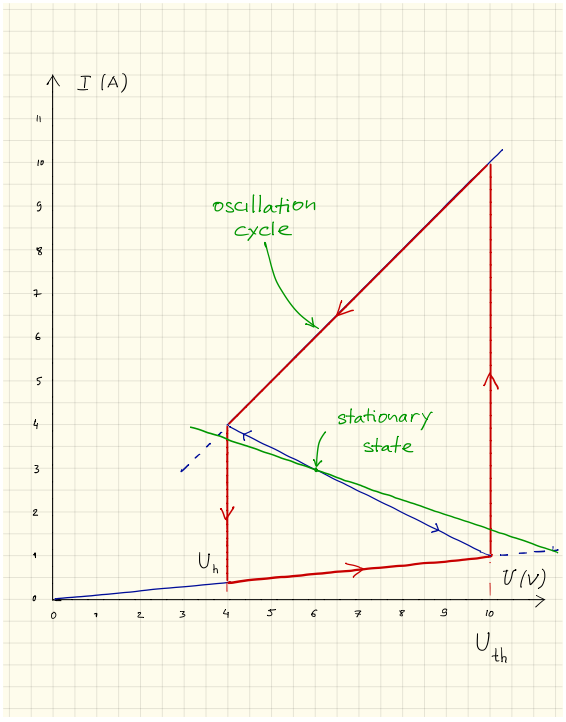
A correctly drawn cycle gives 1.2 points, distributed as follows:

- Switched on branch is part of the cycle **0.2**
- Switched off branch is part of the cycle **0.2**
- Jumps are vertical (constant U) **0.2**
- Jumps are positioned at U_{h} and U_{th} **0.2**
- The system moves to the left on the switched on branch **0.2**

- The system moves to the right on the switched off branch **0.2**

Each of the following observations individually gives up to 0.2 points, but their total cannot exceed 0.6:

- U constant during jumps because the charge on the capacitor cannot change instantaneously **0.2**
- The intermediate branch cannot be part of the cycle because there is a stationary state on it **0.2**
- Jumps occur at corners of the IV graph because at those points the system has nowhere else to go **0.2**
- The system moves moves to the left on the switched on branch because it approaches the stable stationary state (which is located outside the IV graph), or argument with the Kirchoff law **0.2**
- The system moves moves to the right on the switched off branch because it approaches the stable stationary state (which is located outside the IV graph), or argument with the Kirchoff law **0.2**



Solution B2: **[1.9]**

Since the non-linear element is oscillating between the switched on and switched off branches we can put $U_X = R_{\text{on/off}} I_X$. On either of the branches, the circuit behaves as a standard RC-circuit with conductance C and resistance $R_{\text{on/off}} R/(R_{\text{on/off}} + R)$ (the resistor and the element X being connected in parallel). Another way to express it is to **0.5**

write the Kirchhoff law for the switched on and switched off branches

$$R_{\text{on/off}} RC \frac{dI_X}{dt} = \mathcal{E} - (R_{\text{on/off}} + R) I_X$$

The time constant of the circuit is

$$\frac{R_{\text{on/off}} R}{R_{\text{on/off}} + R} C.$$

If the branch in question (switched on or switched off) extended indefinitely, after a long time the system would have landed in a stationary state with the voltage

$$U_{\text{on/off}} = \frac{R_{\text{on/off}}}{R_{\text{on/off}} + R} \mathcal{E}.$$

Then, the time dependence of the voltage drop on the non-linear element is a sum of the constant term $U_{\text{on/off}}$ and of the exponentially decaying term:

$$U_X(t) = \frac{R_{\text{on/off}}}{R_{\text{on/off}} + R} \mathcal{E} + \left(U_{\text{on/off}} - \frac{R_{\text{on/off}}}{R_{\text{on/off}} + R} \mathcal{E} \right) e^{-\frac{R_{\text{on/off}} + R}{R_{\text{on/off}} RC} t}$$

There are 0.5 points distributed as follow for $U_X(t)$:

- Correct exponential **0.2**
- Correct constant term ($t \to \infty$) **0.1**
- Correct coefficient in front of the exponential **0.1**
- Correct equation for $U_X(t)$ **0.1**

Time spent by the system on the switched on branch during one cycle:

$$t_{\text{on}} = \frac{R_{\text{on}} R}{R_{\text{on}} + R} C \log \left(\frac{U_{\text{th}} - U_{\text{on}}}{U_{\text{h}} - U_{\text{on}}} \right) = 2.41 \cdot 10^{-6}\, s, \quad \textbf{0.4}$$

Time spent by the system on the switched off branch during one cycle:

$$t_{\text{off}} = \frac{R_{\text{off}} R}{R_{\text{off}} + R} C \log \left(\frac{U_{\text{off}} - U_{\text{h}}}{U_{\text{off}} - U_{\text{th}}} \right) = 3.71 \cdot 10^{-6}\, s. \quad \textbf{0.4}$$

The total period of oscillations:

$$T = t_{\text{on}} + t_{\text{off}} = 6.12 \cdot 10^{-6}\, s \quad \textbf{0.1}$$

Note: Correct final answers give full points. One may earn points for intermediate steps (see above) for partial answers.

Solution B3: **[0.7]**

Neglect the energy consumed on the switched off branch. The energy consumed

on the switched on branch during the cycle is estimated by

$$E = \frac{1}{R_{\text{on}}} \left(\frac{U_h + U_{th}}{2} \right)^2 t_{\text{on}} = 1.18 \cdot 10^{-4}\,\text{J}. \qquad \textbf{0.4}$$

For the power, this gives an estimate of

$$P \sim \frac{E}{T} = 19.3\,\text{W}. \qquad \textbf{0.3}$$

Note:

- *Formula + answer inside* $5\,\text{W} \leq P \leq 50\,\text{W}$ *give full points*
- *Formula + answer outside the range above but inside* $1\,\text{W} \leq P \leq 100\,\text{W}$ *give 0.5 points*
- *answer outside range but good formula gives 0.4 points*

Also, the proposed formula is only an example, any other reasonable approximation of the integral of the upper branch should be accepted.

Solution B4: **[0.6]**

The wave length of the radio signal is given by $\lambda = cT = 1.82 \cdot 10^3\,\text{m}$. **0.2**
The optimal length of the antenna is $\lambda/4$ (or $3\lambda/4, 5\lambda/4$ *etc.*) **0.3**
The only choice which is below 1 km is $s = \lambda/4 = 459\,\text{m}$. **0.1**
Note: The correct answer $s = \lambda/4 = 459\,\text{m}$ *gives full points, and the mistake* $s = \lambda/2 = 918\,\text{m}$ *only 0.4 pts.*

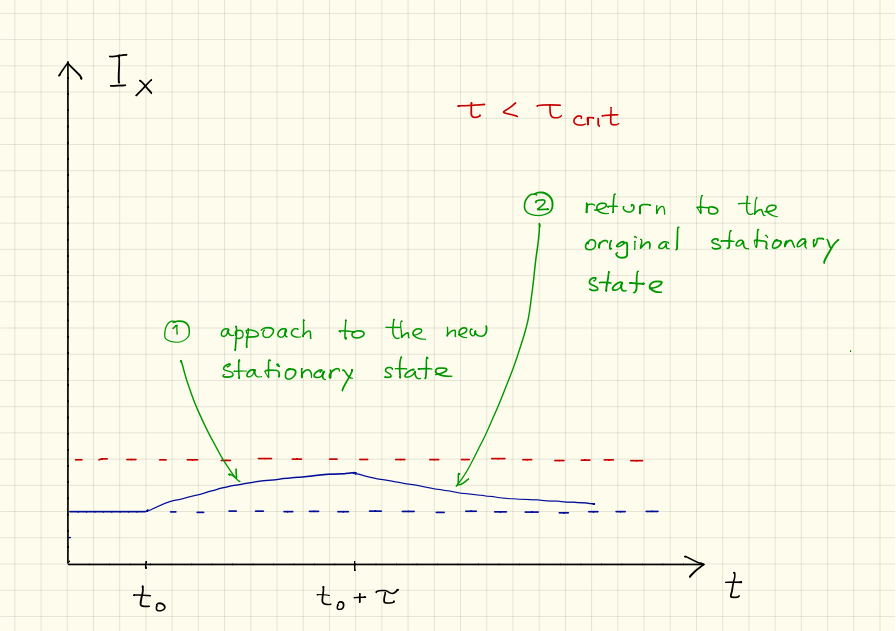
Part C. Bistable non-linear elements in biology: neuristor (2 points)

Solution C1: **[1.2]**

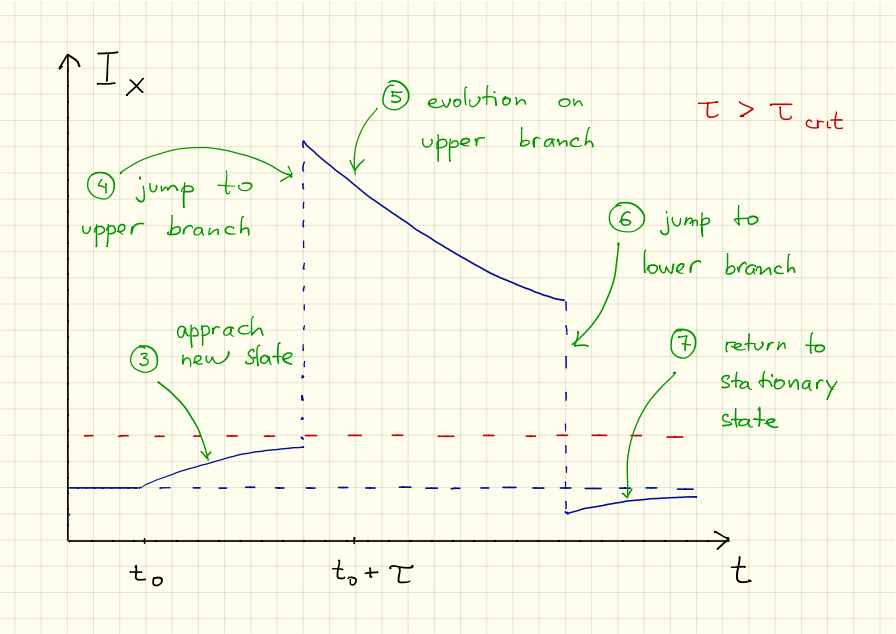
For $\tilde{\mathcal{E}} = 12.0\,\text{V}$, the steady state of the system is located on the switched off branch:

$$\tilde{U} = \frac{R_{\text{off}}}{R + R_{\text{off}}} \tilde{\mathcal{E}} = 9.23\,\text{V}.$$

When the voltage is increased to $\mathcal{E} = 15.0\,\text{V}$, the system starts moving to the right along the switched off branch (in the same way it did in task **B**).
If the voltage drops again before the system reaches the threshold voltage, it will simply return to the stationary state.
If system reaches the threshold voltage, it will jump to the switched on branch, and it will make one oscillations (since $\tau < T$) before the voltage drops again and it returns to the stationary state.



1. Approach to the new stationary state **0.2**
2. Return to the old stationary state **0.2**



3. Approach to the new stationary state **0.1**
4. Jump to the upper branche before $t_0 + \tau$ **0.2**
5. Evolution on the upper branch **0.2**
6. Jump to the lower branche below the old stationary state **0.1**
7. Return to the old stationary state (from below) **0.2**

Solution C2: **[0.6]**

The time needed to reach the threshold voltage is given by

$$\tau_{\text{crit}} = \frac{R_{\text{off}} R}{R_{\text{off}} + R} C \log\left(\frac{U_{\text{off}} - \tilde{U}}{U_{\text{off}} - U_{\text{th}}}\right) = 9.36 \cdot 10^{-7}\, s.$$

Note: This is the same formula as for t_{off} in task **B2**, with U_h replaced by $\tilde{U}$.

- Correct time constant **0.2**
- Correct choice of voltages **0.2**
- Correct final formula **0.1**
- Correct numerical value **0.1**

Note: Correct final answers give full points. One may earn points for intermediate steps (see above) for partial answers.

Solution C3: **[0.2]**

Since $\tau > \tau_{\text{crit}}$, the system will make one oscillation. We conclude that the system is a neuristor. **0.2**

Note: 0.2 are given only if "Yes" is checked, regardless of the development of the other tasks.

Problem 3 : Solution/marking scheme – Large Hadron Collider *(10 points)*

Part A. LHC Accelerator (6 points)

A1 (0.7 pt) Find the exact expression for the final velocity v of the protons as a function of the accelerating voltage V, and fundamental constants.

Solution A1: **[0.7]**

Conservation of energy:

$$m_p \cdot c^2 + V \cdot e = m_p \cdot c^2 \cdot \gamma = \frac{m_p \cdot c^2}{\sqrt{1 - v^2/c^2}} \quad \textbf{0.5}$$

Penalties

No or incorrect total energy **-0.3**

Missing rest mass **-0.2**

Solve for velocity:

$$v = c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{m_p \cdot c^2 + V \cdot e}\right)^2} \quad \textbf{0.2}$$

---

without proton rest mass: **[0.5]**

$$V \cdot e \simeq m_p \cdot c^2 \cdot \gamma = \frac{m_p \cdot c^2}{\sqrt{1 - v^2/c^2}} \quad \textbf{0.3}$$

Solve for velocity:

$$v = c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{V \cdot e}\right)^2} \quad \textbf{0.2}$$

Classical solution: **[0.2]**

$$v = \sqrt{\frac{2 \cdot e \cdot V}{m_p}} \quad \textbf{0.2}$$

A2 (0.8 pt) For particles with high energy and low rest mass the relative deviation $\Delta = (c - v)/c$ of the final velocity v from the speed of light is very small. Find a suitable approximation for Δ and calculate Δ for electrons with an energy of 60.0 GeV.

Solution A2: **[0.8]**

velocity (from previous question):

$$v = c \cdot \sqrt{1 - \left(\frac{m_e \cdot c^2}{m_e \cdot c^2 + V \cdot e}\right)^2} \quad \text{or} \quad c \cdot \sqrt{1 - \left(\frac{m_e \cdot c^2}{V \cdot e}\right)^2} \qquad \textbf{0.1}$$

relative difference:

$$\Delta = \frac{c - v}{c} = 1 - \frac{v}{c} \qquad \textbf{0.1}$$

$$\rightarrow \Delta \simeq \frac{1}{2}\left(\frac{m_e \cdot c^2}{m_e \cdot c^2 + V \cdot e}\right)^2 \quad \text{or} \quad \frac{1}{2}\left(\frac{m_e \cdot c^2}{V \cdot e}\right)^2 \qquad \textbf{0.4}$$

relative difference

$$\Delta = 3.63 \cdot 10^{-11} \qquad \textbf{0.2}$$

---

classical solution gives no points **0.0**

A3 (1.0 pt) Derive an expression for the uniform magnetic flux density B necessary to keep the proton beam on a circular track. The expression should only contain the energy of the protons E, the circumference L, fundamental constants and numbers. You may use suitable approximations if their effect is smaller than the precision given by the least number of significant digits. Calculate the magnetic flux density B for a proton energy of $E = 7.00$ TeV.

Solution A3: **[1.0]**

Balance of forces:

$$\frac{\gamma \cdot m_p \cdot v^2}{r} = \frac{m_p \cdot v^2}{r \cdot \sqrt{1 - \frac{v^2}{c^2}}} = e \cdot v \cdot B \qquad \textbf{0.3}$$

In case of a mistake, partial points can be given for intermediate steps (up to max 0.2). Examples:

Example: Lorentz force **0.1**

Example: $\frac{\gamma \cdot m_p \cdot v^2}{r}$ **0.1**

Energy:

$$E = (\gamma - 1) \cdot m_p \cdot c^2 \simeq \gamma \cdot m_p \cdot c^2 \rightarrow \gamma = \frac{E}{m_p c^2}$$

Therefore:

$$\frac{E \cdot v}{c^2 \cdot r} = e \cdot B \qquad \textbf{0.3}$$

With

$$v \simeq c \quad \text{and} \quad r = \frac{L}{2\pi}$$

follows:

$$\rightarrow B = \frac{2\pi \cdot E}{e \cdot c \cdot L} \qquad \textbf{0.2}$$

Solution:

$$B = 5.50\text{T} \qquad \textbf{0.2}$$

Penalty for < 2 or > 4 significant digits **-0.1**

Calculation without approximations is also correct but does not give more points

$$B = \frac{2\pi \cdot m_p \cdot c}{e \cdot L} \cdot \sqrt{\left(\frac{E}{m_p \cdot c^2}\right)^2 - \left(1 + \frac{m \cdot c^2}{E}\right)^2} \qquad \textbf{0.5}$$

Penalty for each algebraic mistake **-0.1**

Classical calulation gives completely wrong result and maximum 0.3 pt **[0.3]**

$$\frac{m_p \cdot v^2}{r} = e \cdot v \cdot B \qquad \textbf{0.1}$$

$$B = \frac{2\pi}{L \cdot e}\sqrt{2 \cdot m_p \cdot E} \qquad \textbf{0.1}$$

$B = 0.0901\text{T}$	**0.1**
Penalty for < 2 or > 4 significant digits	**-0.1**

A4 (1.0 pt) An accelerated charged particle radiates energy in the form of electromagnetic waves. The radiated power P_{rad} of a charged particle that circulates with a constant angular velocity depends only on its acceleration a, its charge q, the speed of light c and the permittivity of free space ϵ_0. Use a dimensional analysis to find an expression for the radiated power P_{rad}.

Solution A4: **[1.0]**

Ansatz:

$$P_{rad} = a^{\alpha} \cdot q^{\beta} \cdot c^{\gamma} \cdot \epsilon_0^{\delta}$$ **0.2**

Dimensions: [a]=ms^{-2}, [q]=C=As, [c]=ms^{-1},[ϵ_0]=As(Vm)$^{-1}$=A^{2}s^2(Nm^2)$^{-1}$=A^{2}s^4(kgm^3)$^{-1}$

All dimensions correct **0.3**

Three dimensions correct **0.2**

Two dimensions correct **0.1**

---

if dimensions: N and Coulomb [ϵ_0]= C^2(Nm^2)$^{-1}$

$$\frac{\mathrm{m}^{\alpha}}{\mathrm{s}^{2\alpha}} \cdot \mathrm{C}^{\beta} \cdot \frac{\mathrm{m}^{\gamma}}{\mathrm{s}^{\gamma}} \cdot \frac{\mathrm{C}^{2\delta}}{\mathrm{N}^{\delta} \cdot \mathrm{m}^{2\delta}} = \frac{\mathrm{N} \cdot \mathrm{m}}{\mathrm{s}}$$ **0.1**

From this follows:

$$\mathrm{N} :\rightarrow \delta = -1, \quad \mathrm{C} :\rightarrow \beta + 2 \cdot \delta = 0, \quad \mathrm{m} :\rightarrow \alpha + \gamma - 2\delta = 1, \quad \mathrm{s} :\rightarrow 2 \cdot \alpha + \gamma = 1$$ **0.2**

Two equations correct **0.1**

And therefore:

$$\rightarrow \alpha = 2, \beta = 2, \gamma = -3, \delta = -1$$ **0.1**

---

if dimensions: N and As [ϵ_0]=A^{2}s^2(Nm^2)$^{-1}$

$$\frac{\mathrm{m}^{\alpha}}{\mathrm{s}^{2\alpha}} \cdot \mathrm{A}^{\beta} \cdot \mathrm{s}^{\beta} \cdot \frac{\mathrm{m}^{\gamma}}{\mathrm{s}^{\gamma}} \cdot \frac{\mathrm{A}^{2\delta} \cdot \mathrm{s}^{2\delta}}{\mathrm{N}^{\delta} \cdot \mathrm{m}^{2\delta}} = \frac{\mathrm{N} \cdot \mathrm{m}}{\mathrm{s}}$$ **0.1**

From this follows:

$$N :\rightarrow \delta = -1, \quad \mathrm{A} :\rightarrow \beta + 2 \cdot \delta = 0, \quad \mathrm{m} :\rightarrow \alpha + \gamma - 2\delta = 1, \quad \mathrm{s} :\rightarrow -2 \cdot \alpha + \beta - \gamma + 2\delta = -1$$ **0.2**

Two equations correct **0.1**

And therefore:

$$\rightarrow \alpha = 2, \beta = 2, \gamma = -3, \delta = -1$$ **0.1**

---

if dimensions: kg and As [ϵ_0]=A^{2}s^4(kg$\cdot$ m^3)$^{-1}$

$$\frac{\mathrm{m}^{\alpha}}{\mathrm{s}^{2\alpha}} \cdot \mathrm{A}^{\beta} \cdot \mathrm{s}^{\beta} \cdot \frac{\mathrm{m}^{\gamma}}{\mathrm{s}^{\gamma}} \cdot \frac{\mathrm{A}^{2\delta} \cdot \mathrm{s}^{4\delta}}{\mathrm{kg}^{\delta} \cdot \mathrm{m}^{3\delta}} = \frac{\mathrm{kg} \cdot m^2}{s^3}$$ **0.1**

From this follows:

$\text{kg} :\rightarrow \delta = -1, \quad \text{A} :\rightarrow \beta + 2\cdot\delta = 0, \quad \text{m} :\rightarrow \alpha + \gamma - 3\delta = 2, \quad \text{s} :\rightarrow -2\cdot\alpha + \beta - \gamma + 4\delta = -3$ **0.2**

Two equations correct **0.1**

And therefore:

$$\rightarrow \alpha = 2, \beta = 2, \gamma = -3, \delta = -1$$ **0.1**

Radiated Power:

$$P_{rad} \propto \frac{a^2 \cdot q^2}{c^3 \cdot \epsilon_0}$$ **0.1**

Other solutions with other units are possible and are accepted
No solution but realise that unit of charge must vanish $\beta = 2\delta$ **0.2**

A5 (1.0 pt) Calculate the total radiated power P_{tot} of the LHC for a proton energy of $E = 7.00$ TeV (Note table 1). You may use appropriate approximations.

Solution A5: **[1.0]**

Radiated Power:

$$P_{rad} = \frac{\gamma^4 \cdot a^2 \cdot e^2}{6\pi \cdot c^3 \cdot \epsilon_0}$$ **0.1**

Energy:

$$E = (\gamma - 1) m_p \cdot c^2 \quad \text{or equally valid} \quad E \simeq \gamma \cdot m_p \cdot c^2$$ **0.2**

Acceleration:

$$a \simeq \frac{c^2}{r} \quad \text{with} \quad r = \frac{L}{2\pi}$$ **0.2**

Therefore:

$$P_{rad} = (\frac{E}{m_p c^2} + 1)^4 \cdot \frac{e^2 \cdot c}{6\pi\epsilon_0 \cdot r^2} \quad \text{or} \quad (\frac{E}{m_p c^2})^4 \cdot \frac{e^2 \cdot c}{6\pi\epsilon_0 \cdot r^2}$$ **0.3**

(not required $P_{rad} = 7.94 \cdot 10^{-12}$W)

Total radiated power:

$$P_{tot} = 2 \cdot 2808 \cdot 1.15 \cdot 10^{11} \cdot P_{rad} = 5.13\text{kW}$$ **0.2**

penalty for missing factor 2 (for the two beams): **-0.1** **-0.1**
penalty for wrong numbers 2808 and/or $1.15 \cdot 10^{11}$ (numbers come from table 1): **-0.1** **-0.1**

A6 (1.5 pt) Determine the time T that the protons need to pass through this field.

Solution A6: **[1.5]**

2nd Newton's law

$$F = \frac{dp}{dt} \quad \text{leads to} \qquad \textbf{0.2}$$

$$\frac{V \cdot e}{d} = \frac{p_f - p_i}{T} \text{ with } p_i = 0 \qquad \textbf{0.3}$$

Conservation of energy:

$$E_{tot} = m \cdot c^2 + e \cdot V \qquad \textbf{0.2}$$

Since

$$E_{tot}^2 = (m \cdot c^2)^2 + (p_f \cdot c)^2 \qquad \textbf{0.2}$$

$$\rightarrow p_f = \frac{1}{c} \cdot \sqrt{(m \cdot c^2 + e \cdot V)^2 - (m \cdot c^2)^2} = \sqrt{2e \cdot m \cdot V + \left(\frac{e \cdot V}{c}\right)^2} \qquad \textbf{0.2}$$

$$\rightarrow T = \frac{d \cdot p_f}{V \cdot e} = \frac{d}{V \cdot e}\sqrt{2e \cdot m_p \cdot V + \left(\frac{e \cdot V}{c}\right)^2} \qquad \textbf{0.3}$$

$$T = 218\text{ns} \qquad \textbf{0.1}$$

Alternative solution **[1.5]**

2nd Newton's law

$$F = \frac{dp}{dt} \quad \text{leads to} \qquad \textbf{0.2}$$

$$\frac{V \cdot e}{d} = \frac{p_f - p_i}{T} \text{ with } p_i = 0 \qquad \textbf{0.3}$$

velocity from A1 or from conservation of energy

$$v = c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{m_p \cdot c^2 + V \cdot e}\right)^2} \qquad \textbf{0.2}$$

and hence for γ

$$\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}} = 1 + \frac{e \cdot V}{m_p \cdot c^2} \qquad \textbf{0.2}$$

$$\rightarrow p_f = \gamma \cdot m_p \cdot v = \left(1 + \frac{e \cdot V}{m_p \cdot c^2}\right) \cdot m_p \cdot c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{m_p \cdot c^2 + V \cdot e}\right)^2} \qquad \textbf{0.2}$$

$$\rightarrow T = \frac{d \cdot p_f}{V \cdot e} = \frac{d \cdot m_p \cdot c}{V \cdot e} \cdot \sqrt{\left(\frac{m_p \cdot c^2 + e \cdot V}{m_p \cdot c^2}\right)^2 - 1} = \frac{d}{V \cdot e}\sqrt{2e \cdot m_p \cdot V + \left(\frac{e \cdot V}{c}\right)^2} \qquad \textbf{0.3}$$

$$T = 218\text{ns} \qquad \textbf{0.1}$$

Alternative solution: integrate time **[1.5]**

Energy increases linearly with distance x

$$E(x) = \frac{e \cdot V \cdot x}{d} \qquad \textbf{0.2}$$

$$t = \int dt = \int_0^d \frac{dx}{v(x)} \qquad \textbf{0.2}$$

$$v(x) = c \cdot \sqrt{1 - \left(\frac{m_p \cdot c^2}{m_p \cdot c^2 + \frac{e \cdot V \cdot x}{d}}\right)^2} = c \cdot \frac{\sqrt{\left(m_p \cdot c^2 + \frac{e \cdot V \cdot x}{d}\right)^2 - (m_p \cdot c^2)^2}}{m_p \cdot c^2 + \frac{e \cdot V \cdot x}{d}}$$

$$= c \cdot \frac{\sqrt{\left(1 + \frac{e \cdot V \cdot x}{d \cdot m_p \cdot c^2}\right)^2 - 1}}{1 + \frac{e \cdot V \cdot x}{d \cdot m_p \cdot c^2}} \qquad \textbf{0.2}$$

$$\text{Substitution} : \xi = \frac{e \cdot V \cdot x}{d \cdot m_p \cdot c^2} \qquad \frac{d\xi}{dx} = \frac{e \cdot V}{d \cdot m_p \cdot c^2} \qquad \textbf{0.2}$$

$$\rightarrow t = \frac{1}{c} \int_0^b \frac{1 + \xi}{\sqrt{(1+\xi)^2 - 1}} \frac{d \cdot m_p \cdot c^2}{e \cdot V} d\xi \qquad b = \frac{e \cdot V}{m_p \cdot c^2} \qquad \textbf{0.2}$$

$$1 + \xi := \cosh(s) \qquad \frac{d\xi}{ds} = \sinh(s) \qquad \textbf{0.1}$$

$$t = \frac{m_p \cdot c \cdot d}{e \cdot V} \int \frac{\cosh(s) \cdot \sinh(s) ds}{\sqrt{\cosh^2(s) - 1}} = \frac{m_p \cdot c \cdot d}{e \cdot V} [\sinh(s)]_{b_1}^{b_2} \qquad \textbf{0.2}$$

$$\text{with} \quad b_1 = \cosh^{-1}(1), \quad b_2 = \cosh^{-1}\left(1 + \frac{e \cdot V}{m_p \cdot c^2}\right) \qquad \textbf{0.1}$$

$$T = 218\text{ns} \qquad \textbf{0.1}$$

Alternative: differential equation **[1.5]**

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} \qquad \textbf{0.2}$$

$$\rightarrow \frac{V \cdot e}{d} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m \cdot v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{m \cdot a\left(1 - \frac{v^2}{c^2}\right) + m \cdot a\frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \gamma^3 \cdot m \cdot a \qquad \textbf{0.4}$$

$$a = \ddot{s} = \frac{V \cdot e}{d \cdot m}\left(1 - \frac{\dot{s}^2}{c^2}\right)^{\frac{3}{2}} \qquad \textbf{0.3}$$

$$\text{Ansatz} : \; s(t) = \sqrt{i^2 \cdot t^2 + k} - l \;\; \text{with boundary conditions } s(0) = 0, v(0) = 0 \qquad \textbf{0.1}$$

$$\rightarrow s(t) = \frac{c}{V \cdot e}\left(\sqrt{e^2 \cdot V^2 \cdot t^2 + c^2 \cdot m^2 \cdot d^2} - c \cdot m \cdot d\right) \qquad \textbf{0.2}$$

$$s = d \rightarrow T = \frac{d}{V \cdot e}\sqrt{\left(\frac{V \cdot e}{c}\right)^2 + 2V \cdot e \cdot m} \qquad \textbf{0.2}$$

$$T = 218\text{ns} \qquad \textbf{0.1}$$

classical solution: **[0.4]**

$$F = \frac{V \cdot e}{d} \rightarrow \text{acceleration } a = \frac{F}{m_p} = \frac{V \cdot e}{m_p \cdot d} \qquad \textbf{0.1}$$

$$d = \frac{1}{2} \cdot a \cdot T^2 \rightarrow T = \sqrt{\frac{2d}{a}} \qquad \textbf{0.1}$$

And hence for the time

$$T = d \cdot \sqrt{\frac{2 \cdot m_p}{V \cdot e}} \qquad \textbf{0.1}$$

$$T = 194\text{ns} \qquad \textbf{0.1}$$

Part B. Particle identification (4 points)

B1 (0.8 pt) Express the particle rest mass m in terms of the momentum p, the flight length l and the flight time t assuming that the particles with elementary charge e travel with velocity close to c on straight tracks in the ToF detector and that it travels perpendicular to the two detection planes (see Figure 2).

Solution B1: **[0.8]**

with velocity

$$v = \frac{l}{t}$$ **0.1**

relativistic momentum

$$p = \frac{m \cdot v}{\sqrt{1 - \frac{v^2}{c^2}}}$$ **0.2**

gets

$$p = \frac{m \cdot l}{t \cdot \sqrt{1 - \frac{l^2}{t^2 \cdot c^2}}}$$ **0.2**

$\rightarrow$ mass

$$m = \frac{p \cdot t}{l} \cdot \sqrt{1 - \frac{l^2}{t^2 \cdot c^2}} = \frac{p}{l \cdot c} \cdot \sqrt{t^2 \cdot c^2 - l^2}$$ **0.3**

Alternative **[0.8]**

with flight distance: l, flight time t gets:

$$t = \frac{l}{(c \cdot \beta)}$$ **0.1**

relativistic momentum

$$p = \frac{m \cdot \beta \cdot c}{\sqrt{1 - \beta^2}}$$

therefore the velocity:

$$\beta = \frac{p}{\sqrt{m^2 \cdot c^2 + p^2}}$$ **0.2**

insert into the expression for t:

$$t = l \frac{\sqrt{m^2 \cdot c^2 + p^2}}{c \cdot p}$$ **0.2**

$\rightarrow$ mass:

$$m = \sqrt{\left(\frac{p \cdot t}{l}\right)^2 - \left(\frac{p}{c}\right)^2} = \frac{p}{l \cdot c} \sqrt{(t \cdot c)^2 - (l)^2}$$ **0.3**

---

non-relativistic solution: **[0.0]**

flight time: $t = l/v$ velocity:

$$v = \frac{p}{m} \rightarrow t = \frac{l \cdot m}{p} \quad \text{and} \quad m = \frac{p \cdot t}{l}$$

this solution gives no points **0.0**

B2 (0.7 pt) Calculate the minimal length of a ToF detector that allows to safely distinguish a charged kaon from a charged pion given both their momenta are measured to be 1.00 GeV/c. For a good separation it is required that the difference in the time-of-flight is larger than three times the time resolution of the detector. The typical resolution of a ToF detector is 150 ps (1 ps = 10^{-12} s).

Solution B2: **[0.7]**

Flight time difference between kaon and pion

$$\Delta t = 450\text{ps} = 450 \cdot 10^{-12}\text{s}$$ **0.1**

Flight time difference between kaon and pion

$$\Delta t = \frac{l}{cp}(\sqrt{m_\pi^2 \cdot c^2 + p^2} - \sqrt{m_K^2 \cdot c^2 + p^2}) = 450\text{ps} = 450 \cdot 10^{-12}\text{s}$$ **0.2**

$$\rightarrow l = \frac{\Delta t \cdot p}{\sqrt{m_K^2 + p^2/c^2} - \sqrt{m_\pi^2 + p^2/c^2}}$$ **0.2**

$$\sqrt{m_K^2 + p^2/c^2} = 1.115\ \text{GeV}/c^2 \text{and} \sqrt{m_\pi^2 + p^2/c^2} = 1.010\ \text{GeV}/c^2$$

$$l = 450 \cdot 10^{-12} \cdot \frac{1}{1.115 - 1.010}\text{s GeV}c^2/(\text{GeV}c)$$ **0.1**

$$l = 4285.710^{-12}\text{s} \cdot c = 4285.7 \cdot 10^{-12} \cdot 2.998 \cdot 10^8\text{m} = 1.28\text{m}$$ **0.1**

Penalty for < 2 or > 4 significant digits **-0.1**

Non-relativistic solution: **[0.3]**

Flight time difference between kaon and pion

$$\Delta t = \frac{l}{p}(m_K - m_\pi) = 450ps = 450 \cdot 10^{-12}\text{s}$$ **0.1**

length:

$$l = \frac{\Delta tp}{m_K - m_\Pi} = \frac{450 \cdot 10^{-12}\text{s} \cdot 1\text{GeV}/c}{(0.498 - 0.135)\text{GeV}/c^2}$$ **0.1**

$$l = 450 \cdot 10^{-12}/0.363 \cdot cs = 450 \cdot 10^{-12}/0.363 \cdot 2.998 \cdot 10^8\text{m}$$

$$l = 3716 \cdot 10^{-4}\text{m} = 0.372\text{m}$$ **0.1**

Penalty for < 2 or > 4 significant digits **-0.1**

B3 (1.7 pt) Express the particle mass as a function of the magnetic flux density B, the radius R of the ToF tube, fundamental constants and the measured quantities: radius r of the track and time-of-flight t.

Solution B3: **[1.7]**

Particle is travelling perpendicular to the beam line hence the track length is given by the length of the arc
Lorentz force $\rightarrow$ transverse momentum, since there is no longitudinal momentum, the momentum is the same as the transverse momentum
Use formula from B1 to calculate the mass
track length: length of arc

$$l = 2 \cdot r \cdot \text{asin}\frac{R}{2 \cdot r}$$ **0.5**

penalty for just taking a straight track ($l = R$) **-0.4**

partial points for intermediate steps, maximum 0.4
Lorentz force

$$\frac{\gamma \cdot m \cdot v_t^2}{r} = e \cdot v_t \cdot B \rightarrow p_T = r \cdot e \cdot B$$ **0.4**

partial points for intermediate steps, maximum 0.3
longitudinal momentum=0 $\rightarrow p = p_T$ **0.1**

momentum

$$p = e \cdot r \cdot B$$ **0.1**

$$m = \sqrt{\left(\frac{p \cdot t}{l}\right)^2 - \left(\frac{p}{c}\right)^2} = e \cdot r \cdot B \cdot \sqrt{\left(\frac{t}{2r \cdot \text{asin}\frac{R}{2r})}\right)^2 - \left(\frac{1}{c}\right)^2}$$ **0.6**

partial points for intermediate steps, maximum 0.5

Non-relativistic: track length: length of arc **[0.9]**

$$l = 2 \cdot r \cdot \text{asin}\frac{R}{2 \cdot r}$$ **0.5**

penalty for just taking a straight track ($l = R$) **-0.4**

partial points for intermediate steps, maximum 0.4

$$m = \frac{p \cdot t}{l} = \frac{e \cdot r \cdot B \cdot t}{2r \cdot \text{asin}\frac{R}{2r}} = \frac{e \cdot B \cdot t}{2 \cdot \text{asin}\frac{R}{2r}}$$ **0.4**

partial points for intermediate steps, maximum 0.3

B4 (0.8 pt) Identify the four particles by calculating their mass.

Particle	Radius r [m]	Time of flight [ns]
A	5.10	20
B	2.94	14
C	6.06	18
D	2.32	25

Solution B4: **[0.8]**

Particle	arc [m]	p $[\frac{MeV}{c}]$	p $[\frac{mkg}{s}]$ 10^{-19}	pt/l $[\frac{MeVs}{cm}]$ 10^{-6}	pt/l $[\frac{MeV}{c^2}]$	pt/l [kg] 10^{-27}	Mass $[\frac{MeV}{c^2}]$	Mass [kg] 10^{-27}
A	3.786	764.47	4.0855	4.038	1210.6	2.158	938.65	1.673
B	4.002	440.69	2.3552	1.542	462.2	0.824	139.32	0.248
C	3.760	908.37	4.8546	4.349	1303.7	2.324	935.10	1.667
D	4.283	347.76	1.8585	2.030	608.6	1.085	499.44	0.890

Particles A and C are protons, B is a Pion and D a Kaon

correct mass and identification: per particle	**0.2**
penalty for correct mass but no or wrong identification **for 1 or 2 particles**	**-0.1**
penalty for correct mass but no or wrong identification **for 3 or 4 particles**	**-0.2**
wrong mass, correct momentum:**per particle**	**0.1**
wrong momentum, correct arc **for 3 or 4 particles**	**0.2**
wrong momentum, correct arc **for 1 or 2 particles**	**0.1**

---

non relativistic solution $m = pt/l$ Particle identification is not possible **[0.4]**

Particle	arc [m]	p $[\frac{MeV}{c}]$	p $[\frac{mkg}{s}]$ 10^{-19}	$m = p \cdot t/l$ $[\frac{MeVs}{cm}]$ 10^{-6}	$m = p \cdot t/l$ $[\frac{MeV}{c^2}]$	$m = p \cdot t/l$ [kg] 10^{-27}
A	3.786	764.47	4.0855	4.038	1210.6	2.158
B	4.010	440.69	2.3552	1.542	462.2	0.824
C	3.760	908.37	4.8546	4.349	1303.7	2.324
D	4.283	347.76	1.8585	2.030	608.6	1.085

correct mass or correct momentum: **per particle**	**0.1**
wrong momentum, correct arc **for 3 or 4 particles**	**0.2**
wrong momentum, correct arc **for 1 or 2 particles**	**0.1**

Problem 1 : Electrical conductivity in two dimensions - Answer Sheet *(10 points)*

Part A. Four-point-probe (4PP) measurements (1.2 points)

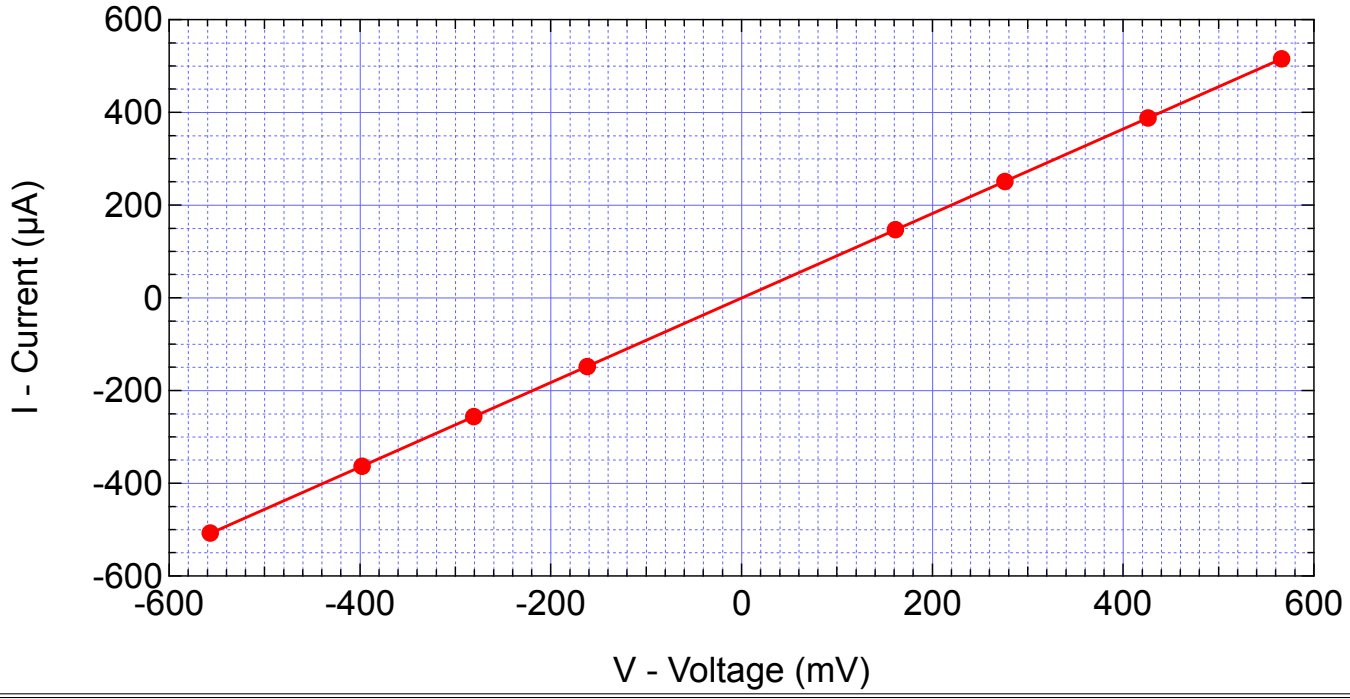
A1 (0.6 pts)

$s = 2$ cm

I (μA)	V (mV)	I (μA)	V (mV)
251	276	-148	-162
516	566	-256	-281
388	426	-363	-398
147	161	-507	-557

Plot your data in the graph B1

Graph B1: I vs V



A2 (0.2 pts)

$R = 1.08$ kΩ

A3 (0.4 pts)

$\Delta R = \pm 1\ \Omega$

Part B. Sheet resistivity (0.3 points)

B1 (0.3 pts)

$\rho_{\square} \equiv \rho_{\infty} = 4.89\ \mathrm{k\Omega}$

Part C. Measurements for different sample dimensions (3.2 points)

C1 (3 pts) and **C2 (0.2 pts)**

$s = 20$ mm

$\rho_\infty = 4.89$ kΩ

w/s	I (μA)	V (mV)	R(w/s) (kΩ)	$R_{average}$ (kΩ)		$\hat{R}$
0.3	92	1477	16.1	15.9		14.7
0.3	74	1184	16			
0.3	57	914	16			
0.3	41	651	15.9			
0.3	23	358	15.6			
0.5	154	1306	8.5	8.5		7.8
0.5	127	1079	8.5			
0.5	97	824	8.5			
0.5	67	567	8.5			
0.5	38	321	8.4			
1	233	1071	4.6	4.6		4.3
1	174	799	4.6			
1	135	621	4.6			
1	101	465	4.6			
1	59	271	4.6			
2.5	389	749	1.9	1.9		1.8
2.5	319	635	2			
2.5	237	457	1.9			
2.5	151	291	1.9			
2.5	74	143	1.9			
5	467	648	1.4	1.4		1.3
5	419	577	1.4			
5	363	499	1.4			
5	289	398	1.4			
5	185	254	1.4			

Part D. Geometrical correction factor (1.9 points)

D1 (1.0 pts)

Plot your data on the appropriate graph paper: linear (Graph E1a), semi-logarithmic (D1b) **or** double-logarithmic (D1c) on the following pages.

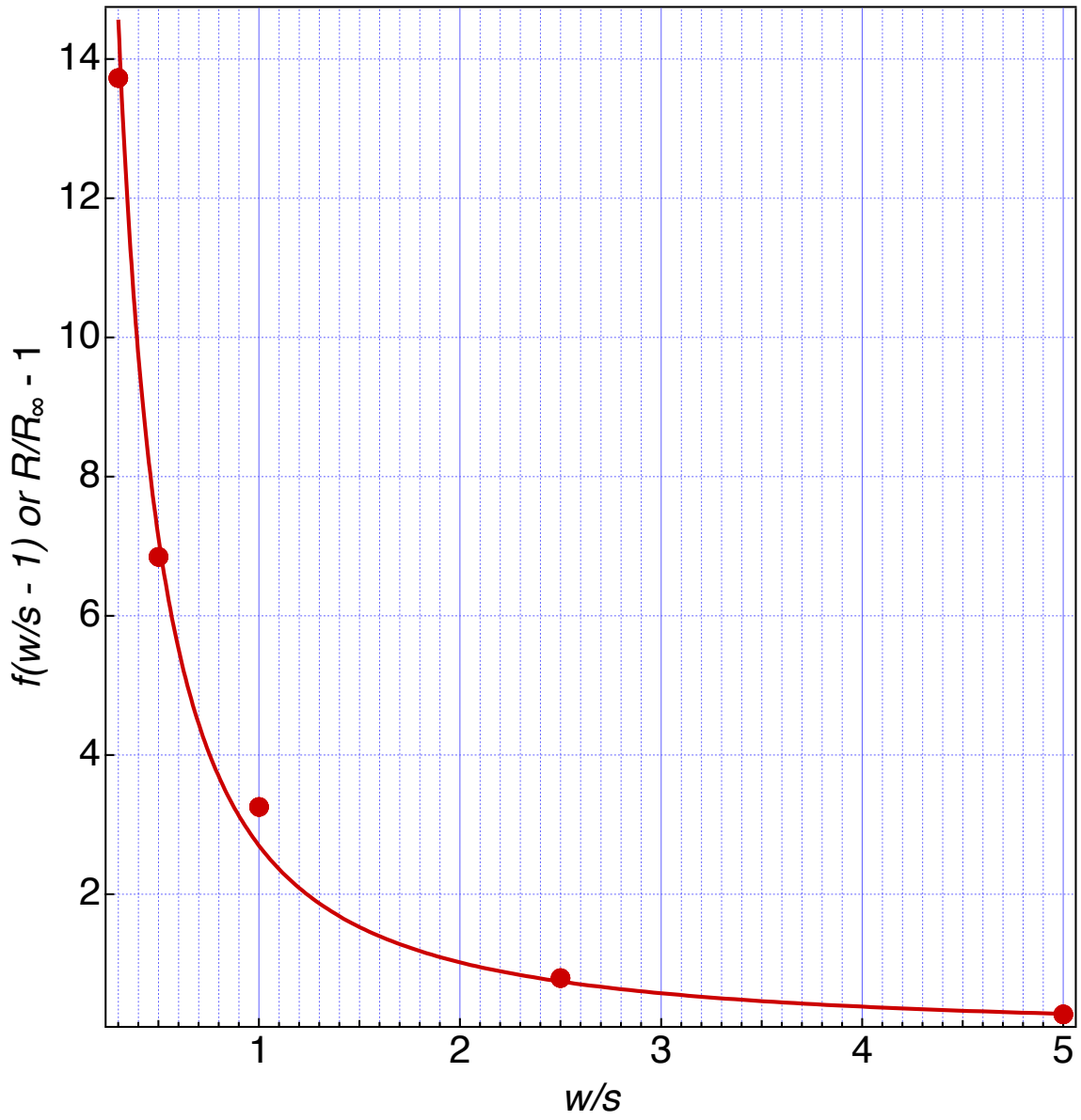
D2 (0.9 pts)

$a = 2.7$

$b = -1.4$

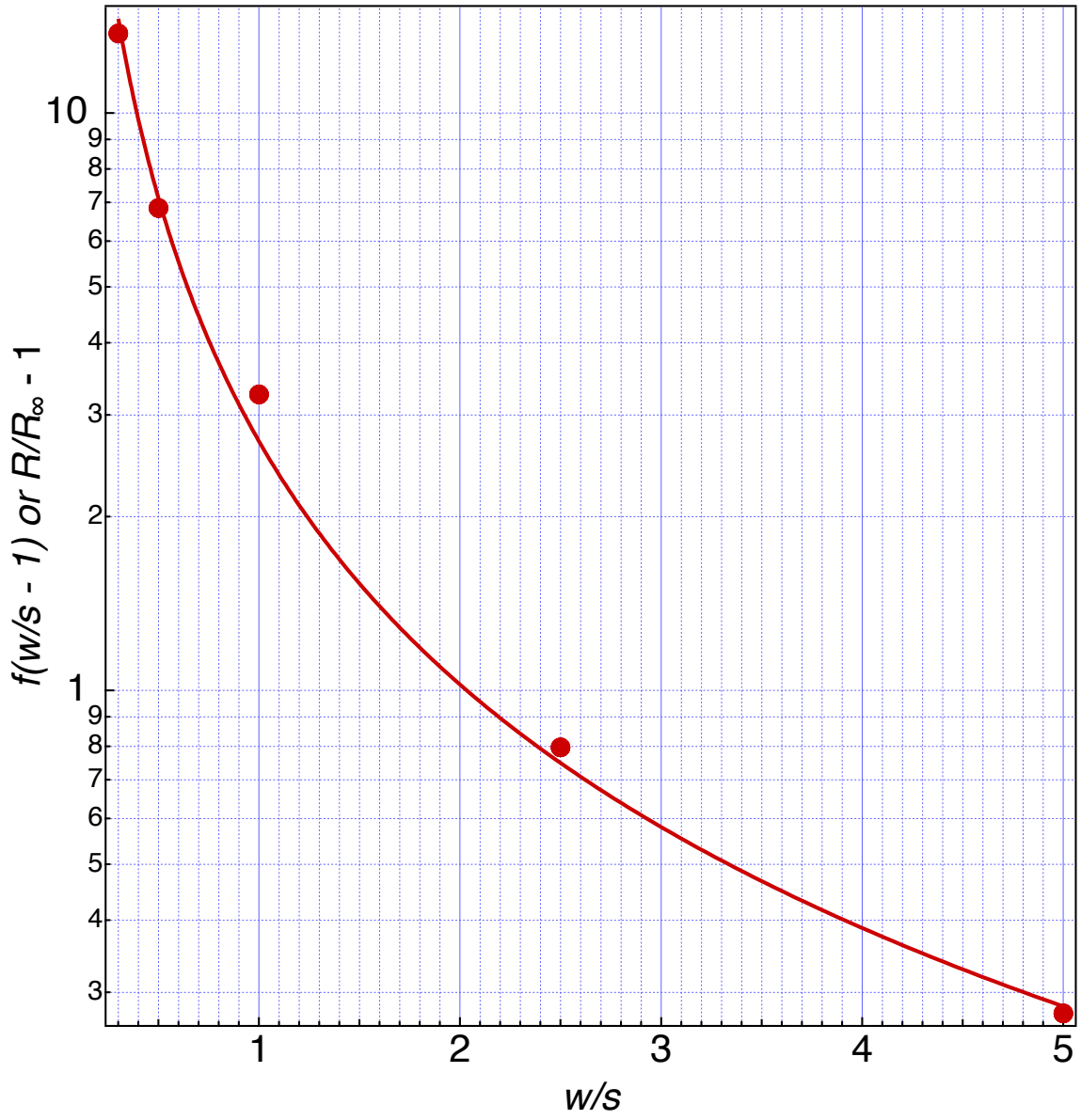
Graph D1a: linear scale: I vs V

Wrong. The usage of linear scale does not allow for deduction of the parameters.



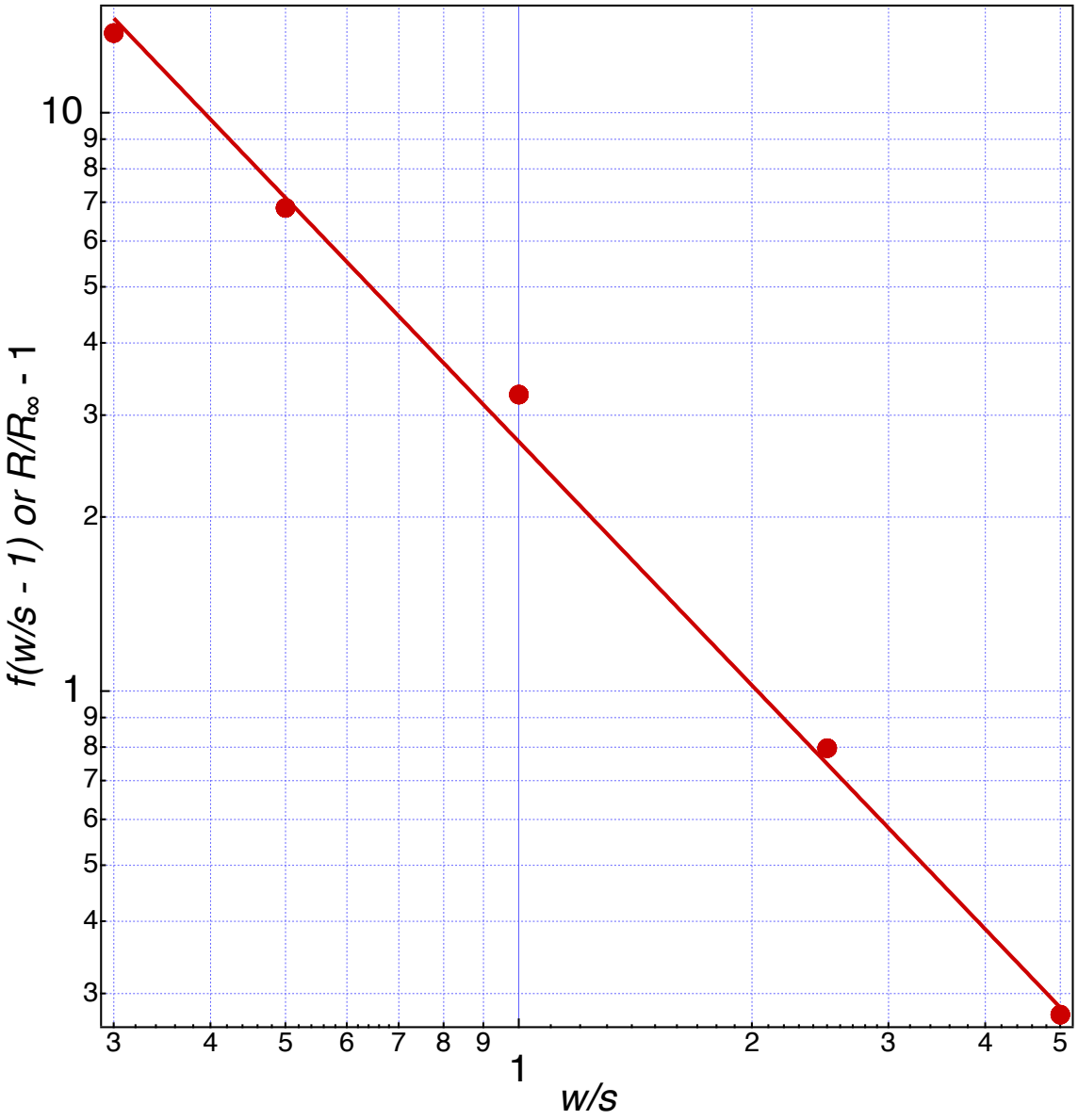
Graph D1b: semi-log scale: I vs V

Wrong. The usage of semi-log scale does not allow for deduction of the parameters.



Graph D1c: double-log scale: I vs V

Correct. The parameters can be deduced by fitting a line.



Part E. van der Pauw-method (3.4 points)

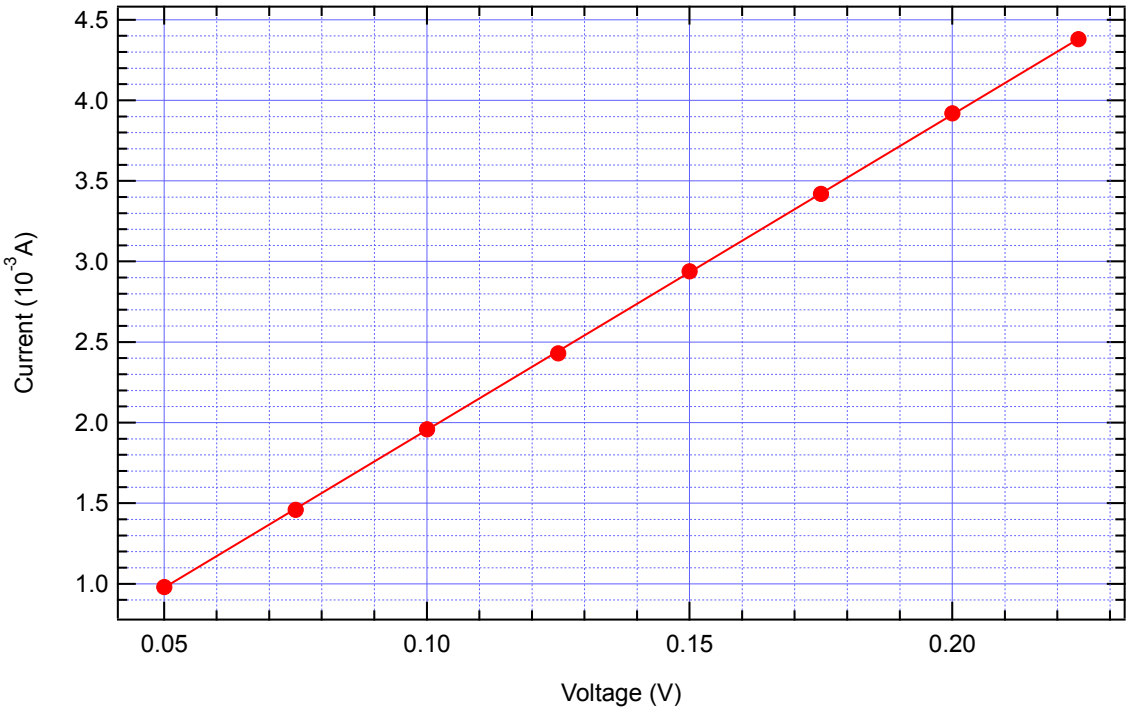
Note the number of your wafer here: 99 (between 1 - 450)

E1 (0.4 pts)

I (mA)	V (mV)	I (mA)	V (mV)
0.98	50	2.94	150
1.46	75	3.42	175
1.96	100	3.92	200
2.43	125	4.38	224

E2 (0.4 pts)

Graph F2: I vs V



$R_{4PP} = 51.1\Omega$

E3 (0.2 pts)

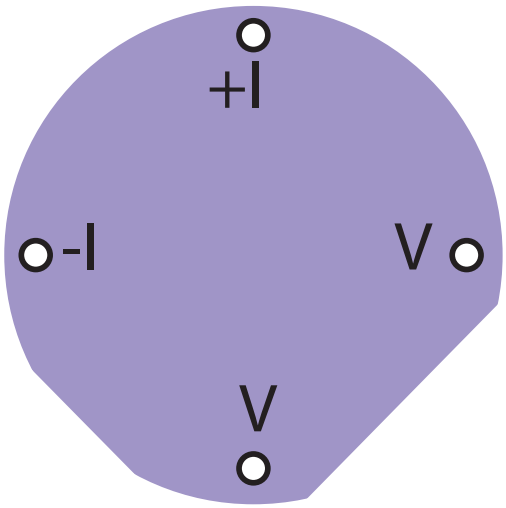
$w = 10$ cm $\rightarrow$ $w/s = 5$ $f(w/s) = 1.284$

E4 (0.1 pts)

$\rho_{\square} = 180\,\Omega$

E5 (0.6 pts)

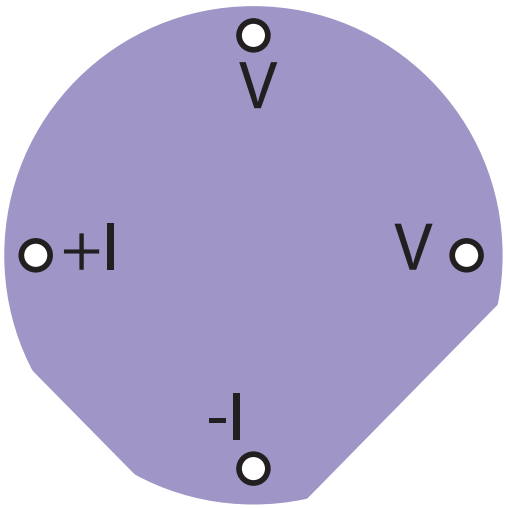
Sketch (orientation of the current):



V mV	I mA
140	3.71
120	3.18
100	2.66
80	2.12
60	1.58
40	1.06
20	0.53
-20	-0.54
-40	-1.06
-60	-1.61
-80	-2.13
-100	-2.68
-120	-3.2
-136	-3.62

E6 (0.6 pts)

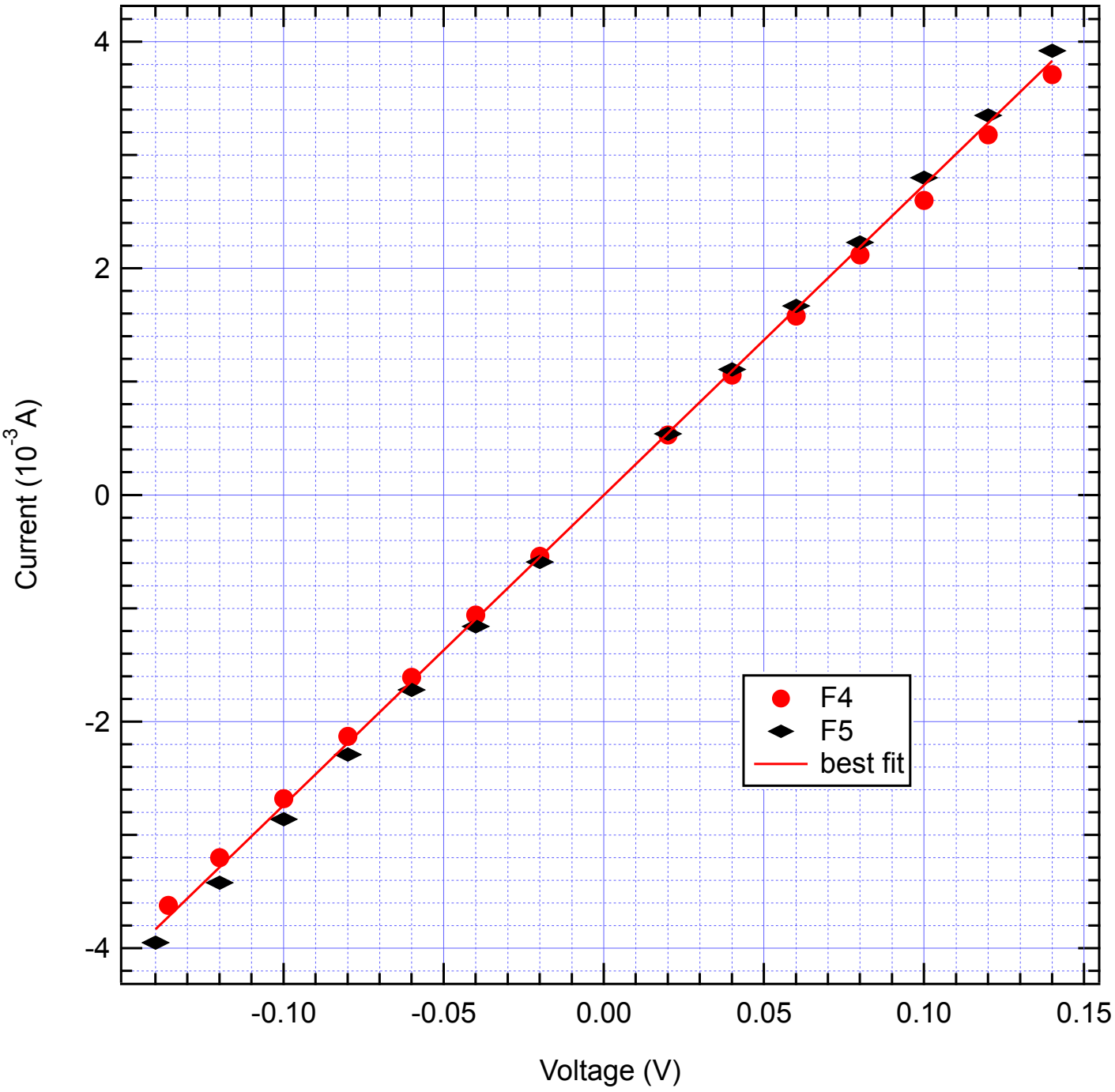
Sketch (orientation of the current):



I	V
140	3.92
120	3.35
100	2.8
80	2.23
60	1.67
40	1.11
20	0.54
-20	-0.59
-40	-1.16
-60	-1.72
-80	-2.29
-100	-2.86
-120	-3.42
-140	-3.95

E7 (0.5 pts)

Graph F6: I vs V



$\langle R \rangle = 36.5\ \Omega$

E8 (0.4 pts) Calculation:

$2 \cdot e^{-\pi \cdot \langle R \rangle / \rho_\square} = 1$ $\quad$ $e^{-\pi \cdot \langle R \rangle / \rho_\square} = 1/2$

$-\dfrac{\pi \cdot \langle R \rangle}{\rho_\square} = \ln(1/2)$ $\quad$ $\dfrac{\pi \cdot \langle R \rangle}{\rho_\square} = \ln(2)$

$\rho_\square = \dfrac{\pi \cdot \langle R \rangle}{\ln(2)}$

$\rho_\square = 165\,\Omega$

E9 (0.1 pts)

$\frac{\Delta \rho_\square}{\rho_\square} =$ 0.091 $=$ 9.1 %

E10 (0.1 pts)

Resistivity of the Cr thin film $\rho = 1.32 \cdot 10^{-6}\,\Omega \cdot \mathrm{m}$

Problem 2 : Solution – Jumping Beads - a model for phase transitions and instabilities *(10 points)*

Part A. Critical driving amplitude (3.3 points)

A1 (1.2 pts)

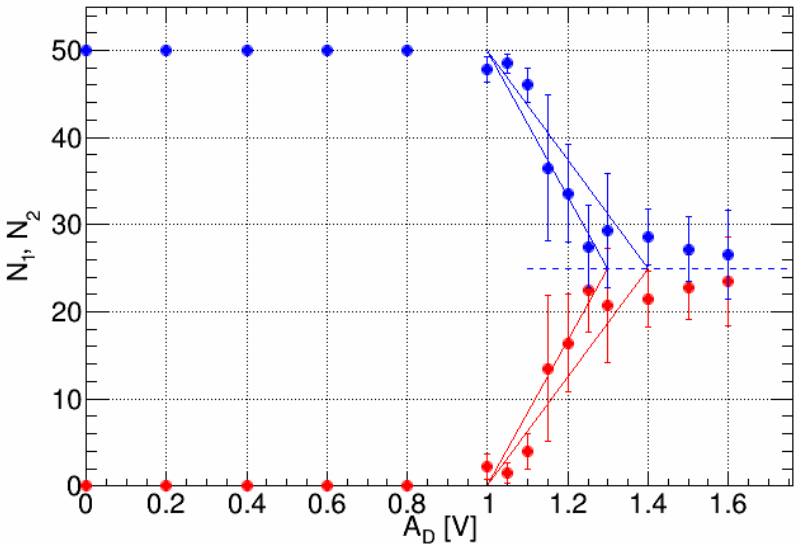
Total number of seeds: $N_0 = 50$.

Number of readings: $n = 6$.

A_D, [V]	N_1						$\bar{N}_1 = \frac{1}{n}\Sigma_{i=1}^{n} N_1^i$	$\bar{N}_2 = N_0 - \bar{N}_1$	$\sigma = \sqrt{\frac{\Sigma_{i=1}^{n}(N_i - \bar{N})^2}{n-1}}$	SE $= \frac{\sigma}{\sqrt{n}}$
1.00	1	5	2	1	2	2	**2.2**	**47.8**	**1.5**	**0.6**
1.05	1	0	2	3	1	2	**1.5**	**48.5**	**1.1**	**0.5**
1.10	4	4	1	7	3	5	**4.0**	**46.0**	**2.0**	**0.8**
1.15	26	5	18	7	18	7	**13.5**	**36.5**	**8.4**	**3.4**
1.20	13	16	27	12	17	13	**16.4**	**33.7**	**5.6**	**2.3**
1.25	26	28	22	22	14	23	**22.5**	**27.5**	**4.8**	**2.0**
1.30	27	24	8	22	22	21	**20.7**	**29.3**	**6.6**	**2.7**
1.40	22	18	17	23	23	25	**21.4**	**28.7**	**3.2**	**1.3**
1.50	19	27	27	24	19	21	**22.8**	**27.2**	**3.7**	**1.5**
1.60	27	15	23	23	23	30	**23.5**	**26.5**	**5.1**	**2.1**

Plot the data in the graph A2.

A2 (1.1 pts)



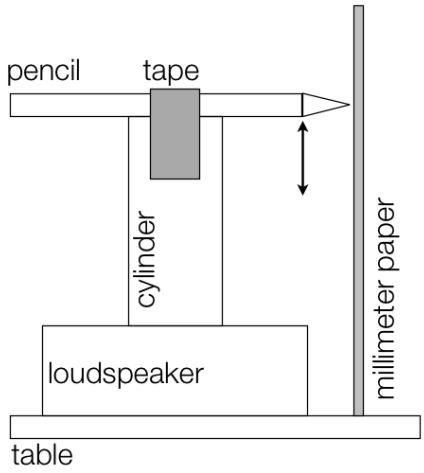
Error bars represent either standard deviation (σ) or standard error (SE).

A3 (1.0 pts)

$A_{D,\text{crit}} = (1.25 \pm 0.05)$ V

Part B. Calibration (3.2 points)

B1 (0.5 pts)

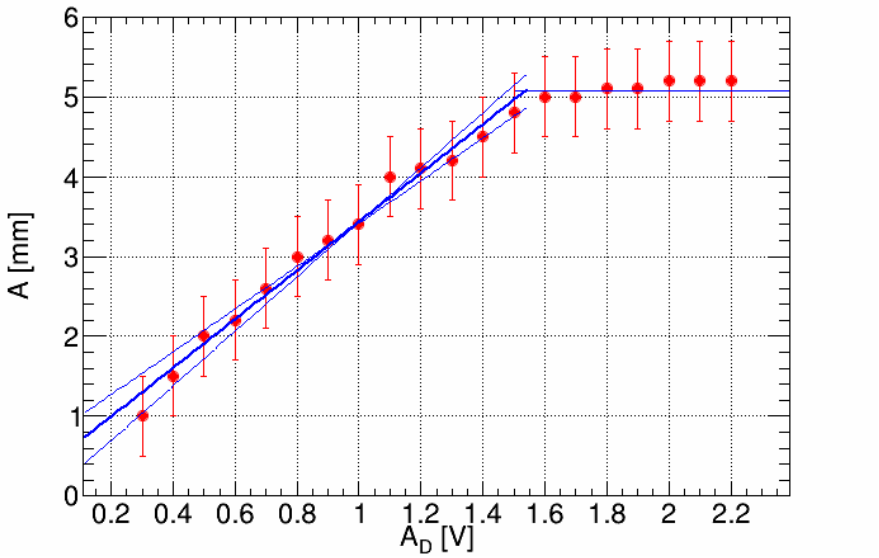


B2 (0.8 pts)

A_D [V]	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1
A [mm]	1.0	1.5	2.0	2.2	2.6	3.0	3.2	3.4	4.0	4.1	4.2	4.5	4.8	5.0	5.0	5.1	5.1	5.2	5.2

Instrumental error ± 0.5 mm.

B3 (1.0 pts)



B4 (0.8 pts)

$A = k_0 + k_1 \times A_D$,
where:
$k_0 = 0.2$ [mm], $k_1 = 3.1$ [mm/V]

B5 (0.1 pts)

$A_{\text{crit}} = (4.4 \pm 0.1)$ mm

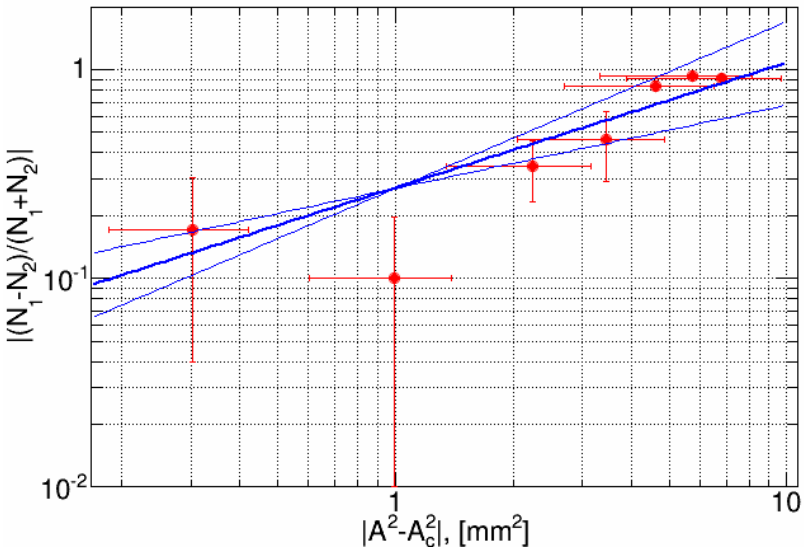
Part C. Critical exponent (3.5 points)

C1 (1.1 pts)

A_D, [V]	A, [mm]	$\lvert\frac{N_1-N_2}{N_1+N_2}\rvert$	$\lvert A^2 - A_0^2\rvert$
1.00	3.5	0.91	6.8
1.05	3.6	0.94	5.7
1.10	3.8	0.84	4.6
1.15	3.9	0.46	3.5
1.20	4.1	0.35	2.2
1.25	4.2	0.10	1.0
1.30	4.4	0.17	0.3
1.40	4.7	0.15	
1.50	5.0	0.09	
1.60	5.3	0.06	

Plot the data in the graph C2.

C2 (1.0 pts)



C3 (1.4 pts)

$y = a \cdot x^b$, where $x = |A^2 - A_0^2|$, $y = |\frac{N_1-N_2}{N_1+N_2}|$.

Critical exponent $b = 0.6 \pm 0.2$.