IPhO
Estonia 2012

The 43rd International Physics Olympiad — Theoretical Competition

Tartu, Estonia — Tuesday, July 17th 2012

- The examination lasts for 5 hours. There are 3 problems worth a total of 30 points. Please note that the point values of the three theoretical problems are not equal.

- **You must not open the envelope with the problems before the sound signal indicating the beginning of competition (three short signals).**

- **You are not allowed to leave your working place without permission.** If you need any assistance (broken calculator, need to visit a restroom, etc), please raise the corresponding flag ("HELP" or "TOILET" with a long handle at your seat) above your seat box walls and keep it raised until an organizer arrives.

- **Your answers must be expressed in terms of those quantities, which are highlighted** in the problem text, and can contain also fundamental constants, if needed. So, if it is written that "the box height is a and the width - b" then a can be used in the answer, and b cannot be used (unless it is highlighted somewhere else, see below). Those quantities which are highlighted in the text of a subquestion can be used only in the answer to that subquestion; the quantities which are highlighted in the introductory text of the Problem (or a Part of a Problem), i.e. outside the scope of any subquestion, can be used for all the answers of that Problem (or of that Problem Part).

- Use only the front side of the sheets of paper.

- For each problem, there are **dedicated Solution Sheets** (see header for the number and pictogramme). Write your solutions onto the appropriate Solution Sheets. For each Problem, the Solution Sheets are numbered; use the sheets according to the enumeration. **Always mark which Problem Part and Question you are dealing with.** Copy the final answers into the appropriate boxes of the **Answer Sheets**. There are also **Draft** papers; use these for writing things which you don't want to be graded. If you have written something what you don't want to be graded onto the Solution Sheets (such as initial and incorrect solutions), cross these out.

- If you need more paper for a certain problem, please raise the flag "HELP" and tell an organizer the problem number; you are given two Solution sheets (you can do this more than once).

- **You should use as little text as possible:** try to explain your solution mainly with equations, numbers, symbols and diagrams.

- The first single sound signal tells you that there are 30 min of solving time left; the second double sound signal means that 5 min is left; the third triple sound signal marks the end of solving time. **After the third sound signal you must stop writing immediately.** Put all the papers into the envelope at your desk. **You are not allowed to take any sheet of paper out of the room.** If you have finished solving before the final sound signal, please raise your flag.

Problem T1. Focus on sketches *(13 points)*

Part A. Ballistics *(4.5 points)*

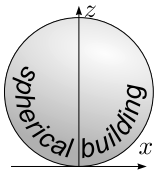
A ball, thrown with an initial speed v_0, moves in a homogeneous gravitational field in the x-z plane, where the x-axis is horizontal, and the z-axis is vertical and antiparallel to the free fall acceleration g. Neglect the effect of air drag.

i. (0.8 pts) By adjusting the launching angle for a ball thrown with a fixed initial speed v_0 from the origin, targets can be hit within the region given by

$$z \leq z_0 - kx^2.$$

You can use this fact without proving it. Find the constants z_0 and k.

ii. (1.2 pts) The launching point can now be freely selected on the ground level $z = 0$, and the launching angle can be adjusted as needed. The aim is to hit the topmost point of a spherical building of radius R (see fig.) with the minimal initial speed v_0. Bouncing off the roof prior to hitting the target is not allowed. Sketch qualitatively the shape of the optimal trajectory of the ball (use the designated box on the answer sheet). Note that the marks are given only for the sketch.



iii. (2.5 pts) What is the minimal launching speed $v_{\min}$ needed to hit the topmost point of a spherical building of radius R?



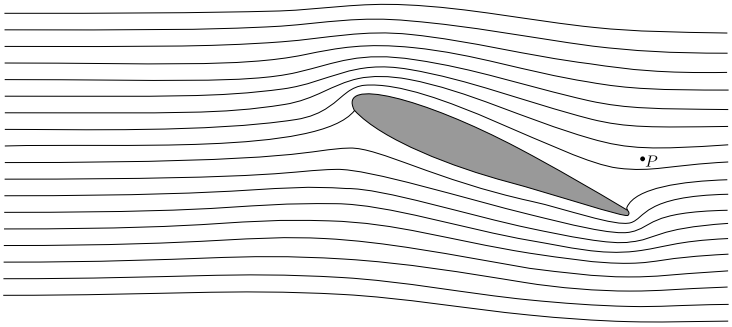
La Geode, Parc de la Villette, Paris. Photo: katchooo/flickr.com

Part B. Air flow around a wing *(4 points)*

For this part of the problem, the following information may be useful. For a flow of liquid or gas in a tube along a streamline, $p + \rho g h + \frac{1}{2}\rho v^2 = \text{const.}$, assuming that the velocity v is much less than the speed of sound. Here ρ is the density, h is the height, g is free fall acceleration and p is hydrostatic pressure. Streamlines are defined as the trajectories of fluid particles (assuming that the flow pattern is stationary). Note that the term $\frac{1}{2}\rho v^2$ is called the dynamic pressure.

In the fig. shown below, a cross-section of an aircraft wing is depicted together with streamlines of the air flow around the wing, as seen in the wing's reference frame. Assume that (a) the air flow is purely two-dimensional (i.e. that the velocity vectors of air lie in the plane of the figure); (b) the streamline pattern is independent of the aircraft speed; (c) there is no wind; (d) the dynamic pressure is much smaller than the atmospheric pressure, $p_0 = 1.0 \times 10^5\,\text{Pa}$.

You can use a ruler to take **measurements from the fig. on the answer sheet**.



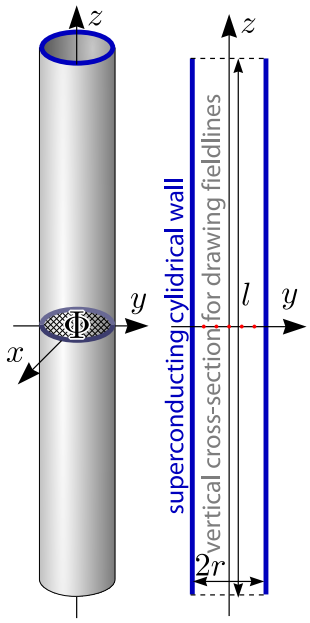
i. (0.8 pts) If the aircraft's ground speed is $v_0 = 100\,\text{m/s}$, what is the speed of the air, v_P, at the point P (marked in the fig.) with respect to the ground?

ii. (1.2 pts) In the case of high relative humidity, as the ground speed of the aircraft increases over a critical value v_{crit}, a stream of water droplets is created behind the wing. The droplets emerge at a certain point Q. Mark the point Q in the fig. on the answer sheet. Explain qualitatively (using formulae and as little text as possible) how you determined the position of Q.

iii. (2.0 pts) Estimate the critical speed v_{crit} using the following data: relative humidity of the air is $r = 90\%$, specific heat capacity of air at constant pressure $c_p = 1.00 \times 10^3\,\text{J/kg}\cdot\text{K}$, pressure of saturated water vapour: $p_{sa} = 2.31\,\text{kPa}$ at the temperature of the unperturbed air $T_a = 293\,\text{K}$ and $p_{sb} = 2.46\,\text{kPa}$ at $T_b = 294\,\text{K}$. Depending on your approximations, you may also need the specific heat capacity of air at constant volume $c_V = 0.717 \times 10^3\,\text{J/kg}\cdot\text{K}$. Note that the relative humidity is defined as the ratio of the vapour pressure to the saturated vapour pressure at the given temperature. Saturated vapour pressure is defined as the vapour pressure by which vapour is in equilibrium with the liquid.

Part C. Magnetic straws *(4.5 points)*

Consider a cylindrical tube made of a superconducting material. The length of the tube is l and the inner radius is r with $l \gg r$. The centre of the tube coincides with the origin, and its axis coincides with the z-axis.

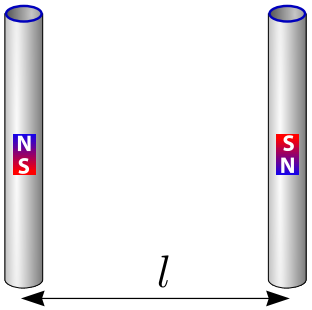


There is a magnetic flux Φ through the central cross-section of the tube, $z = 0$, $x^2 + y^2 < r^2$. A superconductor is a material which expels any magnetic field (the field is zero inside the material).

i. (0.8 pts) Sketch five such magnetic field lines, which pass through the five red dots marked on the axial cross-section of the tube, on the designated diagram on the answer sheet.

ii. (1.2 pts) Find the tension force T along the z-axis in the middle of the tube (i.e. the force by which two halves of the tube, $z > 0$ and $z < 0$, interact with each other).

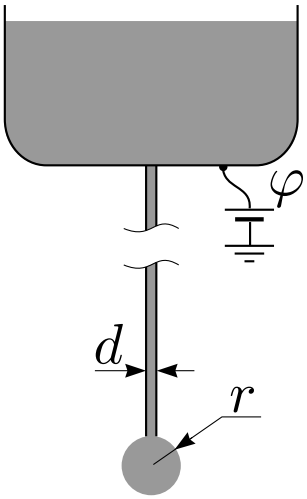
iii. (2.5 pts) Consider another tube, identical and parallel to the first one.



The second tube has the same magnetic field but in the opposite direction and its centre is placed at $y = l, x = z = 0$ (so that the tubes form opposite sides of a square). Determine the magnetic interaction force F between the two tubes.

Problem T2. Kelvin water dropper *(8 points)*

The following facts about the surface tension may turn out to be useful for this problem. For the molecules of a liquid, the positions at the liquid-air interface are less favourable as compared with the positions in the bulk of the liquid. This interface is described by the so-called surface energy, $U = \sigma S$, where S is the surface area of the interface and σ is the surface tension coefficient of the liquid. Moreover, two fragments of the liquid surface pull each other with a force $F = \sigma l$, where l is the length of a straight line separating the fragments.



A long metallic pipe with internal diameter d is pointing directly downwards. Water is slowly dripping from a nozzle at its lower end, see fig. Water can be considered to be electrically conducting; its surface tension is σ and its density is ρ. A droplet of radius r hangs below the nozzle. The radius grows slowly in time until the droplet separates from the nozzle due to the free fall acceleration g. Always assume that $d \ll r$.

Part A. Single pipe *(4 points)*

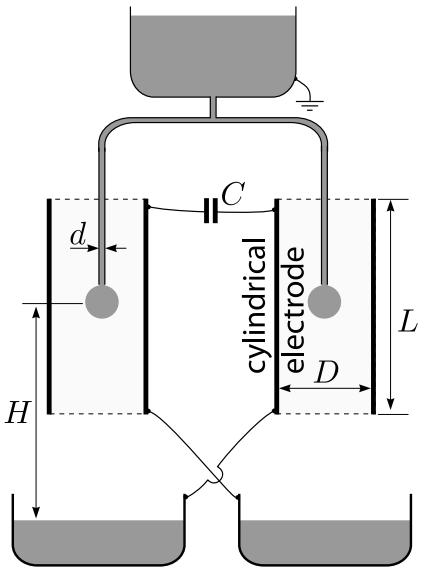
i. (1.2 pts) Find the radius $r_{\max}$ of a drop just before it separates from the nozzle.

ii. (1.2 pts) Relative to the far-away surroundings, the pipe's electrostatic potential is φ. Find the charge Q of a drop when its radius is r.

iii. (1.6 pts) Consider the situation in which r is kept constant and φ is slowly increased. The droplet becomes unstable and breaks into pieces if the hydrostatic pressure inside the droplet becomes smaller than the atmospheric pressure. Find the critical potential $\varphi_{\max}$ at which this will happen.

Part B. Two pipes *(4 points)*

An apparatus called the "Kelvin water dropper" consists of two pipes, each identical to the one described in Part A, connected via a T-junction, see fig. The ends of both pipes are at the centres of two cylindrical electrodes (with height L and diameter D with $L \gg D \gg r$). For both tubes, the dripping rate is n droplets per unit time. Droplets fall from height H into conductive bowls underneath the nozzles, cross-connected to the electrodes as shown in the diagram. The electrodes are connected via a capacitance C. There is no net charge on the system of bowls and electrodes. Note that the top water container is earthed as shown. The first droplet to fall will have some microscopic charge which will cause an imbalance between the two sides and a small charge separation across the capacitor.



i. (1.2 pts) Express the absolute value of the charge Q_0 of the drops as they separate from the tubes, and at the instant when the capacitor's charge is q. Express Q_0 in terms of $r_{\max}$ (from Part A-i) and neglect the effect described in Part A-iii.

ii. (1.5 pts) Find the dependence of q on time t by approximating it with a continuous function $q(t)$ and assuming that $q(0) = q_0$.

iii. (1.3 pts) The dropper's functioning can be hindered by the effect shown in Part A-iii. In addition, a limit $U_{\max}$ to the achievable potential between the electrodes is set by the electrostatic push between a droplet and the bowl beneath it. Find $U_{\max}$.

Problem T3. Protostar formation *(9 points)*

Let us model the formation of a star as follows. A spherical cloud of sparse interstellar gas, initially at rest, starts to collapse due to its own gravity. The initial radius of the ball is r_0 and the mass is m. The temperature of the surroundings (much sparser than the gas) and the initial temperature of the gas is uniformly T_0. The gas may be assumed to be ideal. The average molar mass of the gas is μ and its adiabatic index is $\gamma > \frac{4}{3}$. Assume that $G\frac{m\mu}{r_0} \gg RT_0$, where R is the gas constant and G is the gravitational constant.

i. (0.8 pts) During much of the collapse, the gas is so transparent that any heat generated is immediately radiated away, i.e. the ball stays in thermodynamic equilibrium with its surroundings. What is the number of times, n, by which the pressure increases when the radius is halved to $r_1 = 0.5r_0$? Assume that the gas density remains uniform.

ii. (1 pt) Estimate the time t_2 needed for the radius to shrink from r_0 to $r_2 = 0.95r_0$. Neglect the change of the gravity field at the position of a falling gas particle.

iii. (2.5 pts) Assuming that the pressure remains negligible, find the time $t_{r\to 0}$ needed for the ball to collapse from r_0 down to a much smaller radius, using Kepler's Laws.

iv. (1.7 pts) At some radius $r_3 \ll r_0$, the gas becomes dense enough to be opaque to the heat radiation. Calculate the amount of heat Q radiated away during the collapse from the radius r_0 down to r_3.

v. (1 pt) For radii smaller than r_3 you may neglect heat loss due to radiation. Determine how the temperature T of the ball depends on its radius for $r < r_3$.

vi. (2 pts) Eventually we cannot neglect the effect of the pressure on the dynamics of the gas and the collapse stops at $r = r_4$ (with $r_4 \ll r_3$). However, the radiation loss can still be neglected and the temperature is not yet high enough to ignite nuclear fusion. The pressure of such a protostar is not uniform anymore, but rough estimates with inaccurate numerical prefactors can still be done. *Estimate* the final radius r_4 and the respective temperature T_4.

IPhO
Estonia 2012

The 43rd International Physics Olympiad — Experimental Competition

Tartu, Estonia — Thursday, July 19th 2012

- The examination lasts for 5 hours. There are 2 problems worth in total 20 points. There are two tables in your disposal (in two neighbouring cubicles), the apparatus of Problem E1 is on one table and the apparatus of Problem E2 is on the other table; you can move freely between these tables. **However, you are not allowed to move any piece of experimental setup from one table to the other.**
- Initially the experimental equipment on one table is covered and on the other table is boxed. **You must neither remove the cover nor open the box nor open the envelope with the problems before the sound signal of the beginning of competition (three short signals).**
- **You are not allowed to leave your working place without permission.** If you need any assistance (**malfunctioning equipment**, broken calculator, need to visit a restroom, etc), please raise the corresponding flag ("HELP" or "TOILET" with a long handle at your seat) above your seat box walls and keep it raised until an organizer arrives.
- Use only the front side of the sheets of paper.
- For each problem, there are **dedicated Solution Sheets** (see header for the number and pictogramme). Write your solutions onto the appropriate Solution Sheets. For each Problem, the Solution Sheets are numbered; use the sheets according to the enumeration. Copy the final answers into the appropriate boxes of the **Answer Sheets**. There are also **Draft** papers; use these for writing things which you don't want to be graded. If you have written something what you don't want to be graded onto the Solution Sheets (such as initial and incorrect solutions), cross these out.
- If you need more paper for a certain problem, please raise the flag "HELP" and tell an organizer the problem number; you are given two Solution sheets (you can do this more than once).
- **You should use as little text as possible:** try to explain your solution mainly with equations, numbers, tables, symbols and diagrams.
- Avoid unnecessary movements during the experimental examination and do not shake the walls of your cubicle - the laser experiment requires stability.
- Do not look into the laser beam or its reflections! It may permanently damage your eyes!
- The first single sound signal tells you that there are 30 min of solving time left; the second double sound signal means that 5 min is left; the third triple sound signal marks the end of solving time. **After the third sound signal you must stop writing immediately.** Put all the papers into the envelope at your desk. **You are not allowed to take any sheet of paper out of the room.** If you have finished solving before the final sound signal, please raise your flag.

Problem E1. The magnetic permeability of water *(10 points)*

The effect of a magnetic field on most of substances other than ferromagnetics is rather weak. This is because the energy density of the magnetic field in substances of relative magnetic permeability μ is given by the formula $w = \frac{B^2}{2\mu\mu_0}$, and typically μ is very close to 1. Still, with suitable experimental techniques such effects are firmly observable. In this problem we study the effect of a magnetic field, created by a permanent neodymium magnet, on water and use the results to calculate the magnetic permeability of water. **You are not asked to estimate any uncertainties throughout this problem and you do not need to take into account the effects of surface tension.**

The setup comprises of 1 a **stand** (the highlighted numbers correspond to the numbers in the fig.), 3 a **digital caliper**, 4 a **laser pointer**, 5 a **water tray** and 7 **a cylindrical permanent magnet** in the water tray (the magnet is axially magnetised). The water tray is fixed to the base of the stand by the magnet's pull. The laser is fixed to the caliper, the base of which is fastened to the stand; the caliper allows horizontal displacement of the laser. The on-off button of the laser can be kept down with the help of 13 the white conical tube. Do not leave the Laser switched on unnecessarily. The depth of the water above the magnet should be reasonably close to 1 mm (if shallower, the water surface becomes so curved that it will be difficult to take readings from the screen). 15 A cup of water and 16 a syringe can be used for the water level adjustment (to raise the level by 1 mm, add 13 ml of water). 2 A sheet of graph paper (the "screen") is to be fixed to the vertical plate with 14 small magnetic tablets. If the laser spot on the screen becomes smeared, check for a dust on the water surface (and blow away).

The remaining legend for the figure is as follows: 6 the point where the laser beam hits the screen; 11 the LCD screen of the caliper, 10 the button which switches the caliper units between millimeters and inches; 8 on-off switch; 9 button for setting the origin of the caliper reading. Beneath the laser pointer, there is one more button on the caliper, which temporarily re-sets the origin (if you pushed it inadvertently, push it once again to return to the normal measuring mode).

Numerical values for your calculations:

Horizontal distance between the magnet's centre and the screen $L_0 = 490\,\text{mm}$. Check (and adjust, if needed) the alignment of the centre of the magnet in two perpendicular directions. The vertical axis of the magnet must intersect with the laser beam, and it must also intersect with 12 the black line on the support plate.

Magnetic induction (magnetic field strength) on the magnet's axis, at a height of 1 mm from the flat surface, $B_0 = 0.50\,\text{T}$

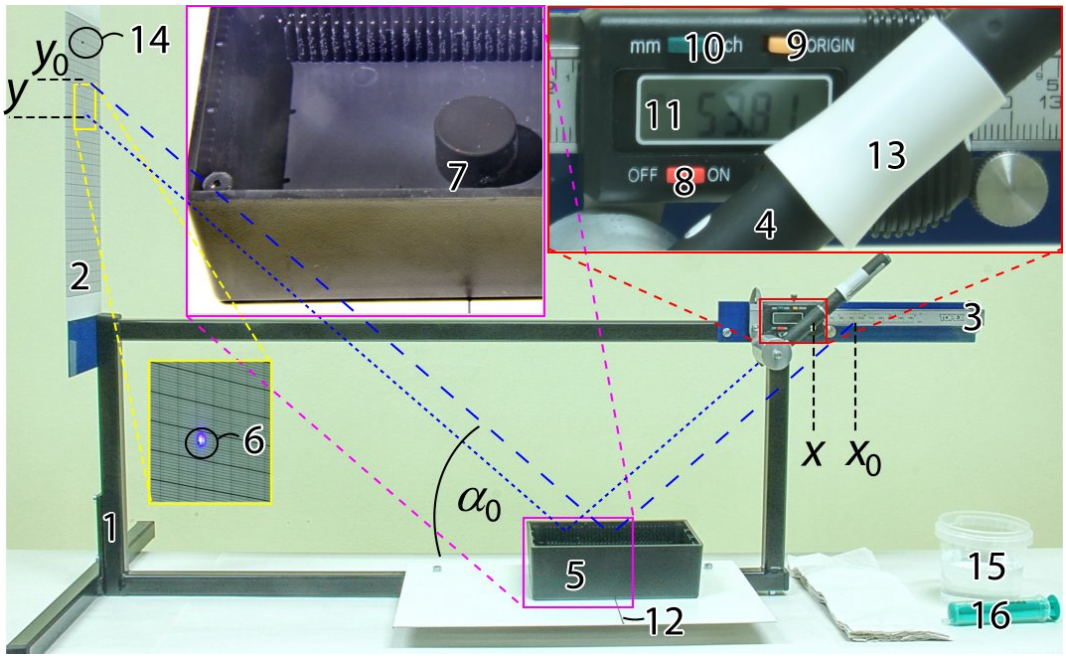
Density of water $\rho_w = 1000\,\text{kg/m}^3$

Acceleration of free-fall $g = 9.8\,\text{m/s}^2$

Permeability of a vacuum $\mu_0 = 4\pi \times 10^{-7}\,\text{H/m}$

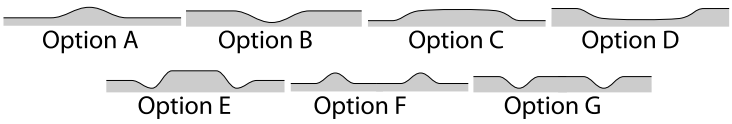
WARNINGS:

- **The laser orientation is pre-adjusted, do not move it!**
- **Do not look into the laser beam or its reflections!**
- **Do not try to remove the strong neodymium magnet!**
- **Do not put magnetic materials close to the magnet!**
- **Turn off the laser when not used, batteries drain in 1 h!**



Part A. Qualitative shape of the water surface *(1 points)*

When a cylindrical magnet is placed below water surface, the latter becomes curved. By observation, determine the shape of the water surface above the magnet. Based on this observation, decide if the water is diamagnetic ($\mu < 1$) or paramagnetic ($\mu > 1$).



Write the letter corresponding to the correct option into the Answer Sheet, together with an inequality $\mu > 1$ or $\mu < 1$.

For this part, you do not need to justify your answer.

Part B. Exact shape of the water surface *(7 points)*

Curving of the water surface can be checked with high sensitivity by measuring the reflection of the laser beam from the surface. We use this effect to calculate the dependence of the depth of the water on the horizontal position above the magnet.

i. (1.6 pts) Measure the dependence of the vertical position y of the laser spot on the screen on the caliper reading x (see figure). You should cover the whole usable range of caliper displacements. Write the results into the Table in the Answer Sheet.

ii. (0.7 pts) Draw the graph of the measured dependence.

iii. (0.7 pts) Using the obtained graph, determine the angle α_0 between the beam and the horizontal surface of the water.

iv. (1.4 pts) please note that the slope ($\tan\beta$) of **the water surface** can be expressed as follows:

$$\tan\beta \approx \beta \approx \frac{\cos^2\alpha_0}{2} \cdot \frac{y - y_0 - (x - x_0)\tan\alpha_0}{L_0 + x - x_0},$$

where y_0 is the vertical position of the laser spot on the screen when the beam is reflected from the water surface at the axis of the magnet, and x_0 is the respective position of the caliper. Calculate the values of the slope of the water surface and enter them into the Table on the Answer Sheet. Please note that it may be possible to simplify your calculations if you substitute some combination of terms in the given expression for the slope with a reading from the last graph.

v. (1.6 pts) Calculate the height of the water surface relative to the surface far from the magnet, as a function of x, and write it into the Table on the Answer Sheet.

vi. (1.0 pts) Draw the graph of the latter dependence. Indicate on it the region where the beam hits the water surface directly above the magnet.

Part C. Magnetic permeability *(2 points)*

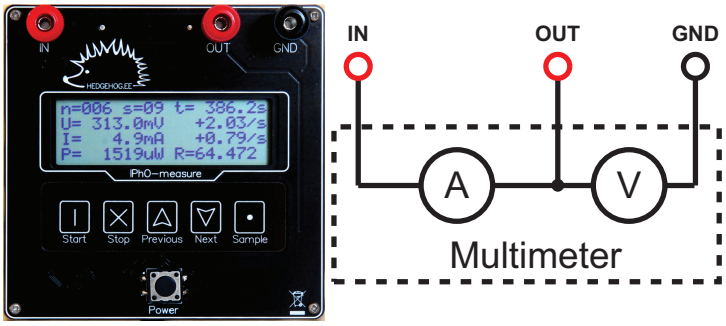
Using the results of Part B, calculate the value of $\mu - 1$ (the so-called *magnetic susceptibility*), where μ is the relative magnetic permeability of the water. Write your final formula and the numerical result into the Answer Sheet.

Problem E2. Nonlinear Black Box *(10 points)*

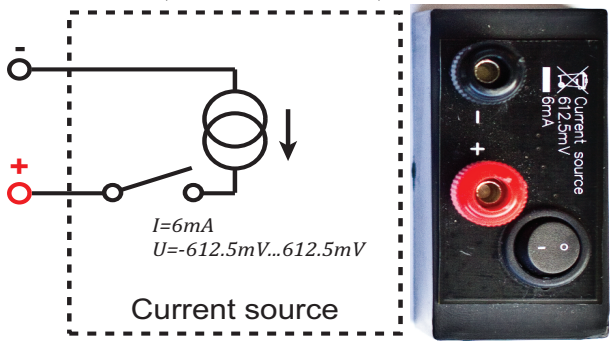
In simple problems, electrical circuits are assumed to consist of linear elements, for which electrical characteristics are directly proportional to each other. Examples include resistance ($V = RI$), capacitance ($Q = CV$) and inductance ($V = L\dot{I} = L\frac{dI}{dt}$), where R, C and L are constants. In this problem, however, we examine a circuit containing nonlinear elements, enclosed in a black box, for which the assumption of proportionality no longer holds.

The setup comprises a **multimeter** (labelled "IPhO-measure"), a black box that acts as a **current source**, a **black box** containing nonlinear elements, and four test leads with stackable connectors for wiring. Be careful not to break the seal on the black box.

The multimeter can measure current and voltage simultaneously. You can store with it up to 2000 data points, each consisting of: voltage V, current I, power $P = IV$, resistance $R = V/I$, voltage time derivative $\dot{V}$ ($= \frac{dV}{dt}$), current time derivative $\dot{I}$ ($= \frac{dI}{dt}$) and time t. See multimeter manual for details. If you go beyond 2000 stored data points, the oldest data will be overwritten.

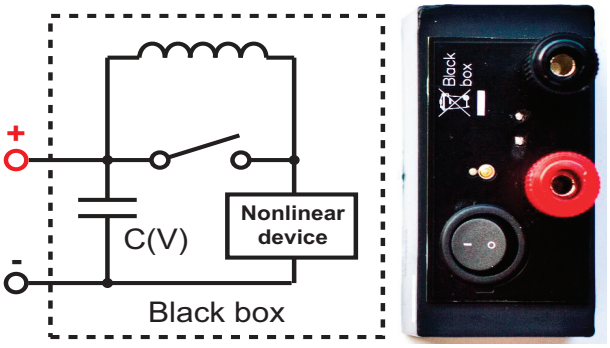


The constant current source supplies stable current as long as the voltage across its terminals stays between $-0.6125\,\mathrm{V}$ and $0.6125\,\mathrm{V}$. When switched off, the constant current source behaves as a large (essentially infinite) resistance.



The black box contains an electric double layer capacitor (which is a slightly nonlinear high capacitance capacitor), an unknown nonlinear element, and an inductor $L = 10\,\mu\mathrm{H}$ of negligible resistance, switchable as indicated on the circuit diagram. The nonlinear element can be considered as a resistance with a nonlinear dependence between the voltage and the current [$I(V)$ is a continuous function of V with $I(0) = 0$]. Likewise, for the capacitor, the differential capacitance $C(V) = dQ/dV$ is not exactly constant.

We say that the voltage on the black box is positive when the potential on its red terminal is higher than the potential on the black terminal. Positive voltage will be acquired when the terminals of matching colours on the black box and the current source are connected (you are allowed to use negative voltages).



It is safe to discharge the capacitor in the black box by shorting its inputs, either by itself or through the IN and OUT terminals on multimeter: the internal resistance of this capacitor is enough to keep the current from damaging anything.

You are not asked to estimate any uncertainties throughout this problem.

Part A. Circuit without inductance *(7 points)*

In this part, keep the switch on the black box closed (push "I" down), so that the inductance is shorted. **Please note** that some measurements may take a considerable time, therefore it is recommended that you read through all the tasks of part A to avoid unnecessary work.

i. (1.0 pts) Confirm that the output current of the current source is approximately 6 mA, and determine the range within which it varies for voltages between 0 and +480 mV. Document the circuit diagram used.

ii. (1.2 pts) Show that the differential capacitance $C(V)$ used in the black box is approximately 2 F by measuring its value for a single voltage of your choice $C(V_0) = C_0$. Document the circuit diagram.

iii. (2.2 pts) Neglecting the nonlinearity of the capacitance [$C(V) \approx C_0$], determine the current–voltage characteristic of the nonlinear element used in the black box. Plot the $I(V)$ curve for obtainable positive voltages on the black box onto the answer sheet. Document the circuit diagram.

iv. (2.6 pts) Using measurements taken from the whole range of obtainable voltages, calculate and plot the $C(V)$ curve for obtainable positive voltages from the black box on the answer sheet. Write down the minimal and maximal values of differential capacitance $C_{\min}, C_{\max}$. Document the circuit diagram.

Part B. Circuit with inductance *(3 points)*

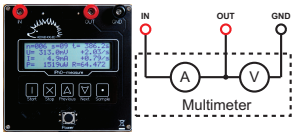
Enable the inductance by opening the switch on the black box (push "0" down). Using the same method as in pt. **A-iii**, measure and plot the current-voltage characteristic of the nonlinear element. Describe any significant differences between the curves of parts A and B and suggest a reason using qualitative arguments. You need to know that the nonlinear element also has a capacitance ($\approx 1\,\mathrm{nF}$) which is connected in parallel to the nonlinear resistance.

IPhO-measure: short manual

IPhO-measure is a multimeter capable of measuring voltage V and current I simultaneously. It also records their time derivatives $\dot{V}$ and $\dot{I}$, their product $P = VI$, ratio $R = V/I$, and time t of the sample. Stored measurements are organized into separate sets; every stored sample is numbered by the set number s and a counter n inside the set. All saved samples are written to an internal flash memory and can later be retrieved.

Electrical behaviour

The device behaves as an ammeter and a voltmeter connected as follows.



	Range	Internal resistance
Voltmeter	0 . . . 2 V	1 MΩ
Voltmeter	2 . . . 10 V	57 kΩ
Ammeter	0 . . . 1 A	1 Ω

Basic usage

- Push "Power" to switch the *IPhO-measure* on. The device is not yet measuring; to start measuring, push "start". Alternatively, you can now start browsing your stored data. See below.
- To browse previously saved samples (through all sets), press "Previous" or "Next". Hold them down longer to jump directly between sets.
- While not measuring, push "Start" to start measuring a new set.
- While measuring, push "Sample" each time you want to store a new set of data (i.e. of the readings shown on the display).
- While measuring, you can also browse other samples of the current set, using "Previous" and "Next".
- Press "Stop" to end a set and stop measuring. The device is still on. You are ready to start a new measuring session or start browsing stored data.
- Pushing "Power" turns the device off. The device will show text "my mind is going ..."; do not worry, all the data measurements will be stored and you will be able to browse them after you switch the device on again. Saved samples will not be erased.

Display



A displayed sample consists of nine variables:

1. index n of the sample in the set;
2. index s of the set;
3. time t since starting the set;
4. voltmeter output V;
5. rate of change of V (the time derivative $\dot{V}$); if derivative cannot be reliably taken due to fluctuations, "`+nan/s`" is shown;
6. ammeter output I;
7. rate of change of I (the time derivative $\dot{I}$); if derivative cannot be reliably taken due to fluctuations, "`+nan/s`" is shown;
8. product $P = VI$;
9. ratio $R = V/I$.

If any of the variables is out of its allowed range, its display shows "`+inf`" or "`-inf`".

PROBLEM

Problem 1

Problem T1. Focus on sketches *(13 points)*

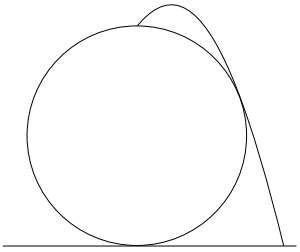
Part A. Ballistics *(4.5 points)*

i. (0.8 pts) When the stone is thrown vertically upwards, it can reach the point $x = 0$, $z = v_0^2/2g$ (as it follows from the energy conservation law). Comparing this with the inequality $z \le z_0 - kx^2$ we conclude that

$$z_0 = v_0^2/2g. \qquad [0.3 \text{ pts}]$$

Let us consider the asymptotics $z \to -\infty$; the trajectory of the stone is a parabola, and at this limit, the horizontal displacement (for the given z) is very sensitive with respect to the curvature of the parabola: the flatter the parabola, the larger the displacement. The parabola has the flattest shape when the stone is thrown horizontally, $x = v_0 t$ and $z = -gt^2/2$, i.e. its trajectory is given by $z = -gx^2/2v_0^2$. Now, let us recall that $z \le z_0 - kx^2$, i.e. $-gx^2/2v_0^2 \le z_0 - kx^2 \Rightarrow k \le g/2v_0^2$. Note that $k < g/2v_0^2$ would imply that there is a gap between the parabolic region $z \le z_0 - kx^2$ and the given trajectory $z = -gx^2/2v_0^2$. This trajectory is supposed to be optimal for hitting targets far below ($z \to -\infty$), so there should be no such a gap, and hence, we can exclude the option $k < g/2v_0^2$. This leaves us with

$$k = g/2v_0^2. \qquad [0.5 \text{ pts}]$$



ii. (1.2 pts) Let us note that the stone trajectory is reversible and due to the energy conservation law, one can equivalently ask, what is the minimal initial speed needed for a stone to be thrown from the topmost point of the spherical building down to the ground without hitting the roof, and what is the respective trajectory. It is easy to understand that the trajectory either needs to touch the roof, or start horizontally from the topmost point with the curvature radius equal to R. Indeed, if neither were the case, it would be possible to keep the same throwing angle and just reduce the speed a little bit — the stone would still reach the ground without hitting the roof. Further, if it were tangent at the topmost point, the trajectory wouldn't touch nor intersect the roof anywhere else, because the curvature of the parabola has maximum at its topmost point. Then, it would be possible to keep the initial speed constant, and increase slightly the throwing angle (from horizontal to slightly upwards): the new trajectory wouldn't be neither tangent at the top nor touch the roof at any other point; now we can reduce the initial speed as we argued previously. So we conclude that the optimal trajectory needs to touch the roof somewhere, as shown in Fig.

iii. (2.5 pts) The brute force approach would be writing down the condition that the optimal trajectory intersects with the building at two points and touches at one. This would be described by a fourth order algebraic equation and therefore, it is not realistic to accomplish such a solution within a reasonable time frame.

Note that the interior of the building needs to lie inside the region where the targets can be hit with a stone thrown from the top with initial speed $v_{\min}$. Indeed, if we can throw over the building, we can hit anything inside by lowering the throwing angle. On the other hand, the boundary of the targetable region needs to touch the building. Indeed, if there were a gap, it would be possible to hit a target just above the point where the optimal trajectory touches the building; the trajectory through that target wouldn't touch the building anywhere, hence we arrive at a contradiction.

So, with v_0 corresponding to the optimal trajectory, the targetable region touches the building; due to symmetry, overall there are two touching points (for smaller speeds, there would be four, and for larger speeds, there would be none). With the origin at the top of the building, the intersection points are defined by the following system of equations:

$$x^2 + z^2 + 2zR = 0, \quad z = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

Upon eliminating z, this becomes a biquadratic equation for x:

$$x^4 \left(\frac{g}{2v_0^2}\right)^2 + x^2 \left(\frac{1}{2} - \frac{gR}{v_0^2}\right) + \left(\frac{v_0^2}{4g} + R\right)\frac{v_0^2}{g} = 0.$$

Hence the speed by which the real-valued solutions disappear can be found from the condition that the discriminant vanishes:

$$\left(\frac{1}{2} - \frac{gR}{v_0^2}\right)^2 = \frac{1}{4} + \frac{gR}{v_0^2} \implies \frac{gR}{v_0^2} = 2.$$

Bearing in mind that due to the energy conservation law, at the ground level the squared speed is increased by $4gR$. Thus we finally obtain

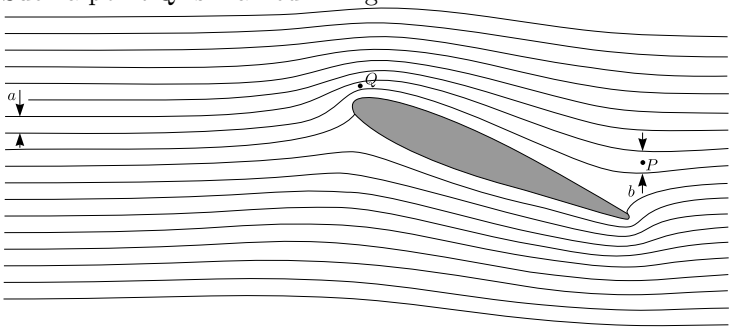
$$v_{\min} = \sqrt{v_0^2 + 4gR} = 3\sqrt{\frac{gR}{2}}.$$

Part B. Mist *(4 points)*

i. (0.8 pts) In the plane's reference frame, along the channel between two streamlines the volume flux of air (volume flow rate) is constant due to continuity. The volume flux is the product of speed and channel's cross-section area, which, due to the two-dimensional geometry, is proportional to the channel width and can be measured from the Fig. Due to the absence of wind, the unperturbed air's speed in the plane's frame is just v_0. So, upon measuring the dimensions $a = 10\,\text{mm}$ and $b = 13\,\text{mm}$ (see Fig), we can write $v_0 a = ub$ and hence $u = v_0 \frac{a}{b}$. Since at point P, the streamlines are horizontal where all the velocities are parallel, the vector addition is reduced to the scalar addition: the air's ground speed $v_P = v_0 - u = v_0(1 - \frac{a}{b}) = 23\,\text{m/s}$.

ii. (1.2 pts) Although the dynamic pressure $\frac{1}{2}\rho v^2$ is relatively small, it gives rise to some adiabatic expansion and compression. In expanding regions the temperature will drop and hence, the pressure of saturated vapours will also drop. If the dew point is reached, a stream of droplets will appear. This process will start in a point where the adiabatic expansion is maximal, i.e. where the hydrostatic pressure is minimal and consequently, as it follows from the Bernoulli's law $p + \frac{1}{2}\rho v^2 = \text{const}$, the dynamic pressure is maximal: in the place where the air speed in

wing's frame is maximal and the streamline distance minimal. Such a point Q is marked in Fig.



iii. (2 pts) First we need to calculate the dew point for the air of given water content (since the relative pressure change will be small, we can ignore the dependence of the dew point on pressure). The water vapour pressure is $p_w = p_{sa} r = 2.08\,\mathrm{kPa}$. The relative change of the pressure of the saturated vapour is small, so we can linearize its temperature dependence:

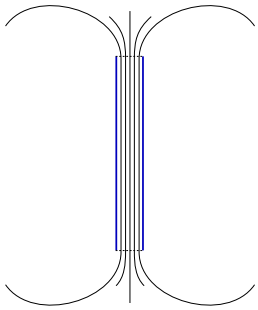
$$\frac{p_{sa}-p_w}{T_a-T} = \frac{p_{sb}-p_{sa}}{T_b-T_a} \implies T_a - T = (T_b-T_a)\frac{(1-r)p_{sa}}{p_{sb}-p_{sa}};$$

numerically $T \approx 291.5\,\mathrm{K}$. Further we need to relate the air speed to the temperature. To this end we need to use the energy conservation law. A convenient ready-to-use form of it is provided by the Bernoulli's law. Applying this law will give a good approximation of the reality, but strictly speaking, it needs to be modified to take into account the compressibility of air and the associated expansion/contraction work. Consider one mole of air, which has the mass μ and the volume $V = RT/p$. Apparently the process is fast and the air parcels are large, so that heat transfer across the air parcels is negligible. Additionally, the process is subsonic; all together we can conclude that the process is adiabatic. Consider a segment of a tube formed by the streamlines. Let us denote the physical quantities at its one end by index 1, and at the other end — by index 2. Then, while one mole of gas flows into the tube at one end, as much flows out at the other end. The inflow carries in kinetic energy $\frac{1}{2}\mu v_1^2$, and the outflow carries out $\frac{1}{2}\mu v_2^2$. The inflowing gas receives work due to the pushing gas equal to $p_1V_1 = RT_1$, the outflowing gas performs work $p_2V_2 = RT_2$. Let's define molar heat capacities $C_V = \mu c_V$ and $C_p = \mu c_p$. The inflow carries in heat energy $C_V RT_1$, and the outflow carries out $C_V RT_2$. All together, the energy balance can be written as $\frac{1}{2}\mu v^2 + C_pT = \text{const}$. From this we can easily express $\Delta\frac{v^2}{2} = \frac{1}{C}v_{\mathrm{crit}}^2(\frac{a^2}{c^2}-1) = c_p\Delta T$, where c is the streamline distance at the point Q, and further

$$v_{\mathrm{crit}} = c\sqrt{\frac{2c_p\Delta T}{a^2-c^2}} \approx 23\,\mathrm{m/s},$$

where we have used $c \approx 4.5\,\mathrm{mm}$ and $\Delta T = 1.5\,\mathrm{K}$. Note that in reality, the required speed is probably somewhat higher, because for a fast condensation, a considerable over-saturation is needed. However, within an order of magnitude, this estimate remains valid.

Part C. Magnetic straws *(4.5 points)*



i. (0.8 pts) Due to the superconducting walls, the magnetic field lines cannot cross the walls, so the flux is constant along the tube. For a closed contour inside the tube, there should be no circulation of the magnetic field, hence the field lines cannot be curved, and the field needs to be homogeneous. The field lines close from outside the tube, similarly to a solenoid.

ii. (1.2 pts) Let us consider the change of the magnetic energy when the tube is stretched (virtually) by a small amount Δl. Note that the magnetic flux trough the tube is conserved: any change of flux would imply a non-zero electromotive force $\frac{d\Phi}{dt}$, and for a zero resistivity, an infinite current. So, the induction $B = \frac{\Phi}{\pi r^2}$. The energy density of the magnetic field is $\frac{B^2}{2\mu_0}$. Thus, the change of the magnetic energy is calculated as

$$\Delta W = \frac{B^2}{2\mu_0}\pi r^2 \Delta l = \frac{\Phi^2}{2\mu_0 \pi r^2}\Delta l.$$

This energy increase is achieved owing to the work done by the stretching force, $\Delta W = T\Delta l$. Hence, the force

$$T = \frac{\Phi^2}{2\mu_0\pi r^2}.$$

iii. (2.5 pts) Let us analyse, what would be the change of the magnetic energy when one of the straws is displaced to a small distance. The magnetic field inside the tubes will remain constant due to the conservation of magnetic flux, but outside, the magnetic field will be changed. The magnetic field outside the straws is defined by the following condition: there is no circulation of $\vec{B}$ (because there are no currents outside the straws); there are no sources of the field lines, other than the endpoints of the straws; each of the endpoints of the straws is a source of streamlines with a fixed magnetic flux $\pm\Phi$. These are exactly the same condition as those which define the electric field of four charges $\pm Q$. We know that if the distance between charges is much larger than the geometrical size of a charge, the charges can be considered as point charges (the electric field near the charges remains almost constant, so that the respective contribution to the change of the overall electric field energy is negligible). Therefore we can conclude that the endpoints of the straws can be considered as magnetic point charges. In order to calculate the force between two magnetic charges (magnetic monopoles), we need to establish the correspondence between magnetic and electric quantities.

For two electric charges Q separated by a distance a, the force is $F = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a^2}$, and at the position of one charge, the electric field of the other charge has energy density $w = \frac{1}{32\pi^2\varepsilon_0}\frac{Q^2}{a^4}$; hence we can write $F = 8\pi w a^2$. This is a universal expression for the force (for the case when the field lines have the same shape as in the case of two opposite and equal by modulus electric charges) relying only on the energy density, and not related to the nature of the field; so we can apply it to the magnetic

field. Indeed, the force can be calculated as a derivative of the full field energy with respect to a virtual displacement of a field line source (electric or magnetic charge); if the energy densities of two fields are respectively equal at one point, they are equal everywhere, and so are equal the full field energies. As it follows from the Gauss law, for a point source of a fixed magnetic flux Φ at a distance a, the induction $B = \frac{1}{4\pi}\frac{\Phi}{a^2}$. So, the energy density $w = \frac{B^2}{2\mu_0} = \frac{1}{32\pi^2\mu_0}\frac{\Phi^2}{a^4}$, hence

$$F = \frac{1}{4\pi\mu_0}\frac{\Phi^2}{a^2}.$$

For the two straws, we have four magnetic charges. The longitudinal (along a straw axis) forces cancel out (the diagonally positioned pairs of same-sign-charges push in opposite directions). The normal force is a superposition of the attraction due to the two pairs of opposite charges, $F_1 = \frac{1}{4\pi\mu_0}\frac{\Phi^2}{l^2}$, and the repulsive forces of diagonal pairs, $F_2 = \frac{\sqrt{2}}{8\pi\mu_0}\frac{\Phi^2}{2l^2}$. The net attractive force will be

$$F = 2(F_1 - F_2) = \frac{4-\sqrt{2}}{8\pi\mu_0}\frac{\Phi^2}{l^2}.$$

Problem T2. Kelvin water dropper *(8 points)*

Part A. Single pipe *(4 points)*

i. (1.2 pts) Let us write the force balance for the droplet. Since $d \ll r$, we can neglect the force $\frac{\pi}{4}\Delta p d^2$ due to the excess pressure Δp inside the tube. So, the gravity force $\frac{4}{3}\pi r_{\max}^3 \rho g$ is balanced by the capillary force. When the droplet separates from the tube, the water surface forms in the vicinity of the nozzle a "neck", which has vertical tangent. In the horizontal cross-section of that "neck", the capillary force is vertical and can be calculated as $\pi\sigma d$. So,

$$r_{\max} = \sqrt[3]{\frac{3\sigma d}{4\rho g}}.$$

ii. (1.2 pts) Since $d \ll r$, we can neglect the change of the droplet's capacitance due to the tube. On the one hand, the droplet's potential is φ; on the other hand, it is $\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$. So,

$$Q = 4\pi\varepsilon_0 \varphi r.$$

iii. (1.6 pts) Excess pressure inside the droplet is caused by the capillary pressure $2\sigma/r$ (increases the inside pressure), and by the electrostatic pressure $\frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\varphi^2/r^2$ (decreases the pressure). So, the sign of the excess pressure will change, if $\frac{1}{2}\varepsilon_0\varphi_{\max}^2/r^2 = 2\sigma/r$, hence

$$\varphi_{\max} = 2\sqrt{\sigma r/\varepsilon_0}.$$

The expression for the electrostatic pressure used above can be derived as follows. The electrostatic force acting on a surface charge of density σ and surface area S is given by $F = \sigma S \cdot \bar{E}$, where $\bar{E}$ is the field at the site without the field created by the surface charge element itself. Note that this force is perpendicular to the surface, so F/S can be interpreted as a pressure. The surface charge gives rise to a field drop on the surface equal to $\Delta E = \sigma/\varepsilon_0$ (which follows from the Gauss law); inside the droplet, there is no field due to the conductivity of the droplet: $\bar{E} - \frac{1}{2}\Delta E = 0$; outside the droplet, there is field $E = \bar{E} + \frac{1}{2}\Delta E$, therefore $\bar{E} = \frac{1}{2}E = \frac{1}{2}\Delta E$. Bringing everything together, we obtain the expression used above.

Note that alternatively, this expression can be derived by considering a virtual displacement of a capacitor's surface and comparing the pressure work $p\Delta V$ with the change of the electrostatic field energy $\frac{1}{2}\varepsilon_0 E^2 \Delta V$.

Finally, the answer to the question can be also derived from the requirement that the mechanical work dA done for an infinitesimal droplet inflation needs to be zero. From the energy conservation law, $dW + dW_{\text{el}} = \sigma\, d(4\pi r^2) + \frac{1}{2}\varphi_{\max}^2\, dC_d$, where the droplet's capacitance $C_d = 4\pi\varepsilon_0 r$; the electrical work $dW_{\text{el}} = \varphi_{\max} dq = 4\pi\varepsilon_0\varphi_{\max}^2 dr$. Putting $dW = 0$ we obtain an equation for $\varphi_{\max}$, which recovers the earlier result.

Part B. Two pipes *(4 points)*

i. (1.2 pts) This is basically the same as Part A-ii, except that the surroundings' potential is that of the surrounding electrode, $-U/2$ (where $U = q/C$ is the capacitor's voltage) and droplet has the ground potential (0). As it is not defined which electrode is the positive one, opposite sign of the potential may be chosen, if done consistently. Note that since the cylindrical electrode is long, it shields effectively the environment's (ground, wall, etc) potential. So, relative to its surroundings, the droplet's potential is $U/2$. Using the result of Part A we obtain

$$Q = 2\pi\varepsilon_0 U r_{\max} = 2\pi\varepsilon_0 q r_{\max}/C.$$

ii. (1.5 pts) The sign of the droplet's charge is the same as that of the capacitor's opposite plate (which is connected to the farther electrode). So, when the droplet falls into the bowl, it will increase the capacitor's charge by Q:

$$dq = 2\pi\varepsilon_0 U r_{\max} dN = 2\pi\varepsilon_0 r_{\max} n dt \frac{q}{C},$$

where $dN = n dt$ is the number of droplets which fall during the time dt This is a simple linear differential equation which is solved easily to obtain

$$q = q_0 e^{\gamma t}, \quad \gamma = \frac{2\pi\varepsilon_0 r_{\max} n}{C} = \frac{\pi\varepsilon_0 n}{C}\sqrt[3]{\frac{6\sigma d}{\rho g}}.$$

iii. (1.3 pts) The droplets can reach the bowls if their mechanical energy mgH (where m is the droplet's mass) is large enough to overcome the electrostatic push: The droplet starts at the point where the electric potential is 0, which is the sum of the potential $U/2$, due to the electrode, and of its self-generated potential $-U/2$. Its motion is not affected by the self-generated field, so it needs to fall from the potential $U/2$ down to the potential $-U/2$, resulting in the change of the electrostatic energy equal to $UQ \le mgH$, where $Q = 2\pi\varepsilon_0 U r_{\max}$ (see above). So,

$$U_{\max} = \frac{mgH}{2\pi\varepsilon_0 U_{\max} r_{\max}},$$

$$\therefore U_{\max} = \sqrt{\frac{H\sigma d}{2\varepsilon_0 r_{\max}}} = \sqrt[6]{\frac{H^3 g \sigma^2 \rho d^2}{6\varepsilon_0^3}}.$$

Problem T3. Protostar formation *(9 points)*

i. (0.8 pts)

$$T = \text{const} \implies pV = \text{const}$$
$$V \propto r^3$$
$$\therefore p \propto r^{-3} \implies \frac{p(r_1)}{p(r_0)} = 2^3 = 8.$$

ii. (1 pt) During the period considered the pressure is negligible. Therefore the gas is in free fall. By Gauss' theorem and symmetry, the gravitational field at any point in the ball is equivalent to the one generated when all the mass closer to the center is compressed into the center. Moreover, while the ball has not yet shrunk much, the field strength on its surface does not change much either. The acceleration of the outermost layer stays approximately constant. Thus,

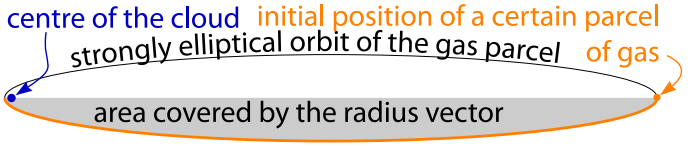
$$t \approx \sqrt{\frac{2(r_0 - r_2)}{g}}$$

where

$$g \approx \frac{Gm}{r_0^2},$$
$$\therefore t \approx \sqrt{\frac{2r_0^2(r_0 - r_2)}{Gm}} = \sqrt{\frac{0.1r_0^3}{Gm}}.$$

iii. (2.5 pts) Gravitationally the outer layer of the ball is influenced by the rest just as the rest were compressed into a point mass. Therefore we have Keplerian motion: the fall of any part of the outer layer consists in a halfperiod of an ultra-elliptical orbit. The ellipse is degenerate into a line; its foci are at the ends of the line; one focus is at the center of the ball (by Kepler's 1st law) and the other one is at r_0, see figure (instead of a degenerate ellipse, a strongly elliptical ellipse is depicted). The period of the orbit is determined by the longer semiaxis of the ellipse (by Kepler's 3rd law). The longer semiaxis is $r_0/2$ and we are interested in half a period. Thus, the answer is equal to the halfperiod of a circular orbit of radius $r_0/2$:

$$\left(\frac{2\pi}{2t_{r\to 0}}\right)^2 \frac{r_0}{2} = \frac{Gm}{(r_0/2)^2} \implies t_{r\to 0} = \pi\sqrt{\frac{r_0^3}{8Gm}}.$$



Alternatively, one may write the energy conservation law $\frac{\dot r^2}{2} - \frac{Gm}{r} = E$ (that in turn is obtainable from Newton's II law $\ddot r = -\frac{Gm}{r^2}$) with $E = -\frac{Gm}{r_0}$, separate the variables ($\frac{dr}{dt} = -\sqrt{2E + \frac{2Gm}{r}}$) and write the integral $t = -\int \frac{dr}{\sqrt{2E+\frac{2Gm}{r}}}$. This integral is probably not calculable during the limitted time given during the Olympiad, but a possible approach can be sketched as follows. Substituting $\sqrt{2E + \frac{2Gm}{r}} = \xi$ and $\sqrt{2E} = \upsilon$, one gets

$$\frac{t_\infty}{4Gm} = \int_0^\infty \frac{d\xi}{(\upsilon^2 - \xi^2)^2}$$
$$= \frac{1}{4\upsilon^3}\int_0^\infty \left[\frac{\upsilon}{(\upsilon-\xi)^2} + \frac{\upsilon}{(\upsilon+\xi)^2} + \frac{1}{\upsilon-\xi} + \frac{1}{\upsilon+\xi}\right] d\xi.$$

Here (after shifting the variable) one can use $\int \frac{d\xi}{\xi} = \ln\xi$ and $\int \frac{d\xi}{\xi^2} = -\frac{1}{\xi}$, finally getting the same answer as by Kepler's laws.

iv. (1.7 pts) By Clapeyron–Mendeleyev law,

$$p = \frac{mRT_0}{\mu V}.$$

Work done by gravity to compress the ball is

$$W = -\int p\,dV = -\frac{mRT_0}{\mu}\int_{\frac{4}{3}\pi r_0^3}^{\frac{4}{3}\pi r_3^3} \frac{dV}{V} = \frac{3mRT_0}{\mu}\ln\frac{r_0}{r_3}.$$

The temperature stays constant, so the internal energy does not change; hence, according to the 1st law of thermodynamics, the compression work W is the heat radiated.

v. (1 pt) The collapse continues adiabatically.

$$pV^\gamma = \text{const} \implies TV^{\gamma-1} = \text{const}.$$
$$\therefore T \propto V^{1-\gamma} \propto r^{3-3\gamma}$$
$$\therefore T = T_0\left(\frac{r_3}{r}\right)^{3\gamma-3}.$$

vi. (2 pts) During the collapse, the gravitational energy is converted into heat. Since $r_3 \gg r_4$, The released gravitational energy can be estimated as $\Delta\Pi = -Gm^2(r_4^{-1} - r_3^{-1}) \approx -Gm^2/r_4$ (exact calculation by integration adds a prefactor $\frac{3}{5}$); the terminal heat energy is estimated as $\Delta Q = c_V\frac{m}{\mu}(T_4 - T_0) \approx c_V\frac{m}{\mu}T_4$ (the approximation $T_4 \gg T_0$ follows from the result of the previous question, when combined with $r_3 \gg r_4$). So, $\Delta Q = \frac{R}{\gamma-1}\frac{m}{\mu}T_4 \approx \frac{m}{\mu}RT_4$. For the temperature T_4, we can use the result of the previous question, $T_4 = T_0\left(\frac{r_3}{r_4}\right)^{3\gamma-3}$. Since initial full energy was approximately zero, $\Delta Q + \Delta\Pi \approx 0$, we obtain

$$\frac{Gm^2}{r_4} \approx \frac{m}{\mu}RT_0\left(\frac{r_3}{r_4}\right)^{3\gamma-3} \implies r_4 \approx r_3\left(\frac{RT_0r_3}{\mu mG}\right)^{\frac{1}{3\gamma-4}}.$$

Therefore,

$$T_4 \approx T_0\left(\frac{RT_0r_3}{\mu mG}\right)^{\frac{3\gamma-3}{4-3\gamma}}.$$

Alternatively, one can obtain the result by approximately equating the hydrostatic pressure $\rho r_4\frac{Gm}{r_4^2}$ to the gas pressure $p_4 = \frac{\rho}{\mu}RT_4$; the result will be exactly the same as given above.

ANSWER SHEET

Problem 1

Problem T1. Focus on sketches *(13 points)*

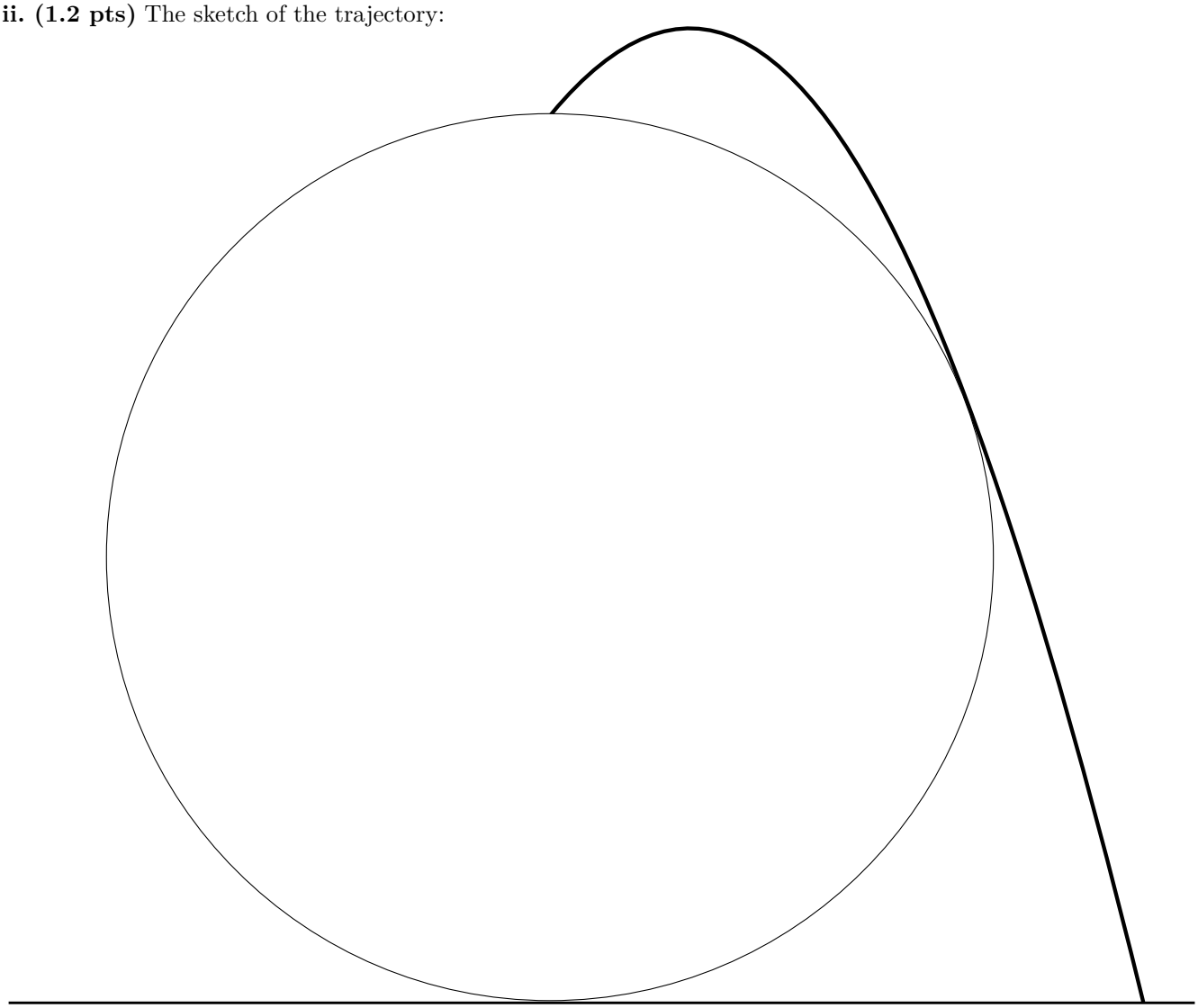
Part A. Ballistics *(4.5 points)*

i. (0.8 pts)

$z_0 = v_0^2/2g$

$k = g/2v_0^2$

ii. (1.2 pts) The sketch of the trajectory:



iii. (2.5 pts)

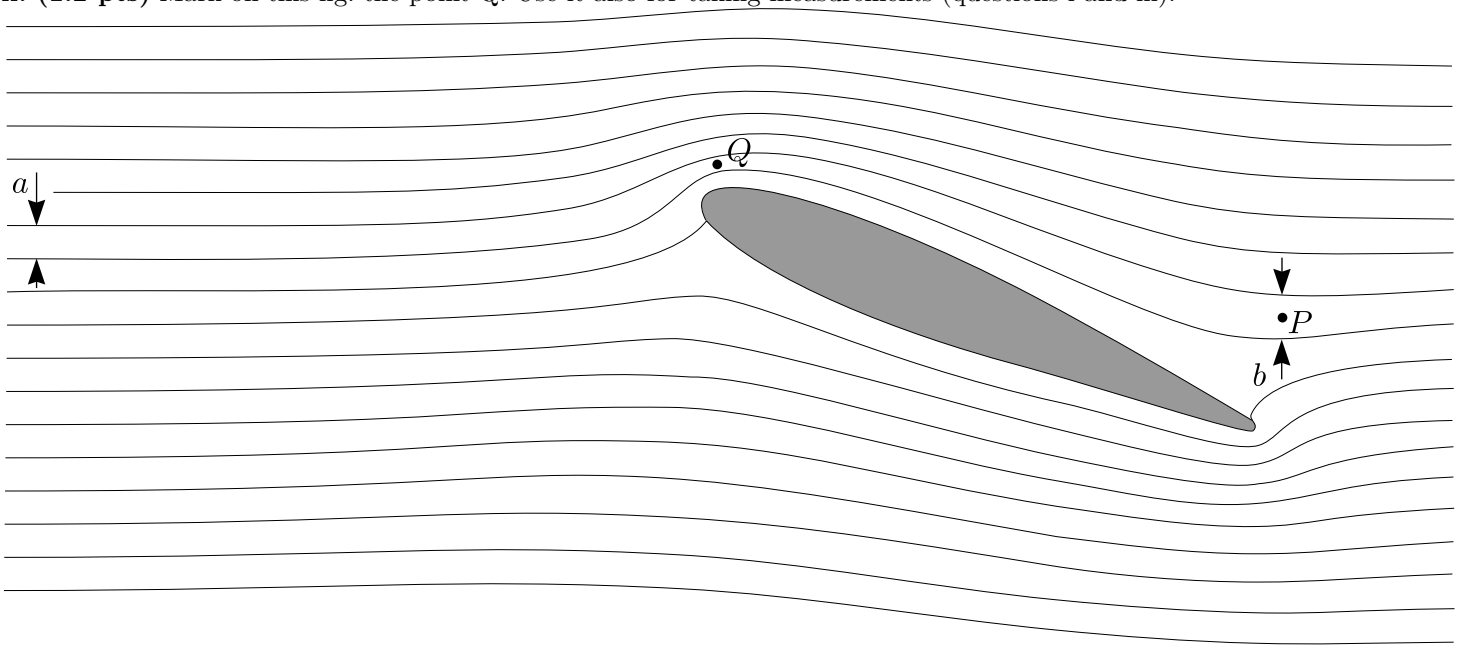
$v_{\min} = 3\sqrt{\frac{gR}{2}}$

Part B. Air flow around a wing *(4 points)*

i. (0.8 pts)

$v_P = 23\,\mathrm{m/s}$

ii. (1.2 pts) Mark on this fig. the point Q. Use it also for taking measurements (questions i and iii).



Formulae motivating the choice of point Q:

$av = \text{const}$

$p + \frac{1}{2}\rho v^2 = \text{const}$

$p^{1-\gamma}T^{\gamma} = \text{const}$

iii. (2.0 pts)

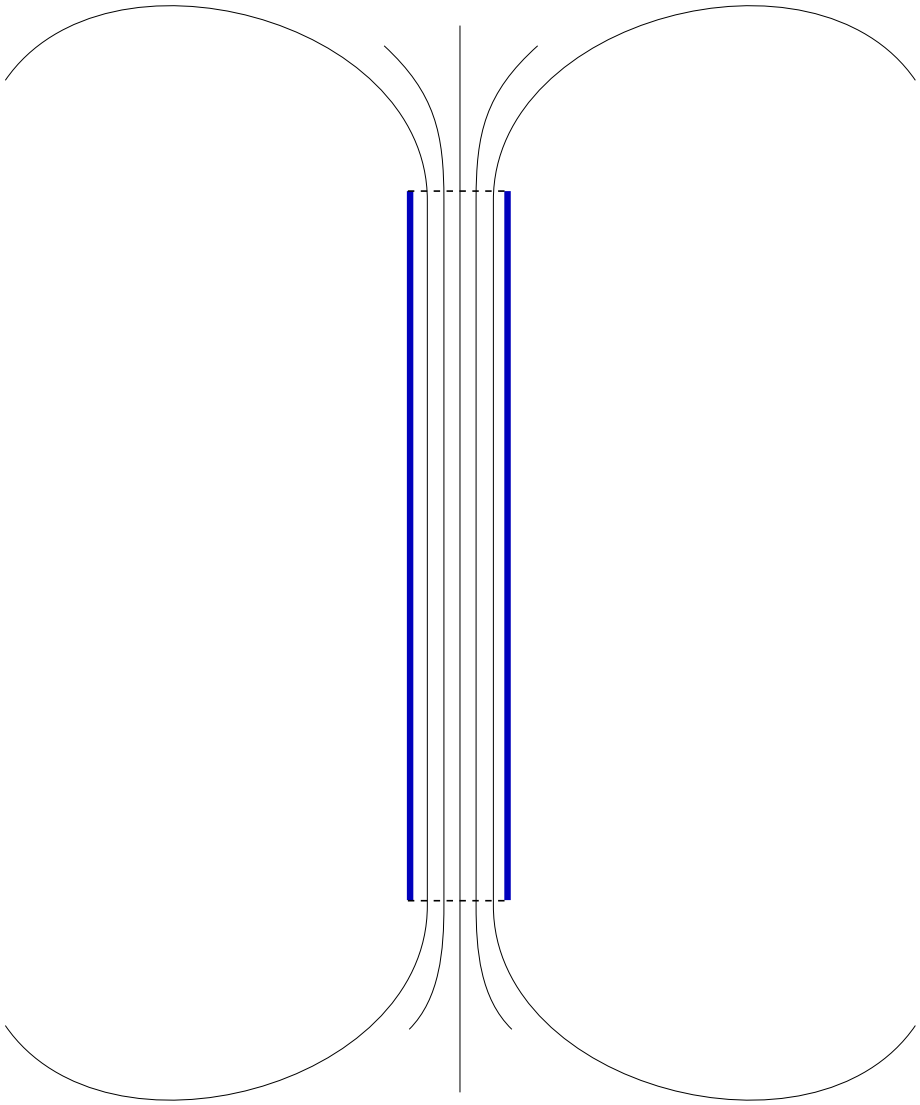
Formula: $v_{\text{crit}} = c\sqrt{\dfrac{2c_p\Delta T}{a^2 - c^2}}$

Numerical: $v_{\text{crit}} \approx 23\,\mathrm{m/s}$

Part C. Magnetic straws *(4.5 points)*

i. (0.8 pts)

Sketch here five magnetic field lines.



ii. (1.2 pts)

$$T = \frac{\Phi^2}{2\mu_0 \pi r^2}$$

iii. (2.5 pts)

$$F = \frac{4-\sqrt{2}}{8\pi\mu_0}\frac{\Phi^2}{l^2}$$

ANSWER SHEET

Problem 2

IPhO Estonia 2012

Problem T2. Kelvin water dropper *(8 points)*

Part A. Single pipe *(4 points)*

i. (1.2 pts)

$$r_{\max} = \sqrt[3]{\frac{3\sigma d}{4\rho g}}$$

ii. (1.2 pts)

$$Q = 4\pi\varepsilon_0\varphi r$$

iii. (1.6 pts)

$$\varphi_{\max} = 2\sqrt{\sigma r/\varepsilon_0}$$

Part B. Two pipes *(4 points)*

i. (1.2 pts)

$$Q_0 = 2\pi\varepsilon_0 q r_{\max}/C$$

ii. (1.5 pts)

$$q(t) = q_0 e^{\gamma t}, \quad \gamma = \frac{\pi\varepsilon_0 n}{C}\sqrt[3]{\frac{6\sigma d}{\rho g}}.$$

iii. (1.3 pts)

$$U_{\max} = \sqrt[6]{\frac{H^3 g\sigma^2\rho d^2}{6\varepsilon_0^3}}$$

Problem T3. Protostar formation *(9 points)*

i. (0.8 pts)

$$n = 8$$

ii. (1 pt)

$$t_2 \approx \sqrt{\frac{0.1 r_0^3}{Gm}}$$

iii. (2.5 pts)

$$t_{r\to 0} = \pi\sqrt{\frac{r_0^3}{8Gm}}$$

iv. (1.7 pts)

$$Q = \frac{3mRT_0}{\mu}\ln\frac{r_0}{r_3}$$

v. (1 pt)

$$T(r) = T_0\left(\frac{r_3}{r}\right)^{3\gamma-3}$$

vi. (2 pts)

$$r_4 \approx r_3\left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{1}{3\gamma-4}}$$

$$T_4 \approx T_0\left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{3\gamma-3}{4-3\gamma}}$$

Problem E1. The magnetic permeability of water *(10 points)*

Part A. Qualitative shape of the water surface *(1 points)*

Observing reflections from the water surface (in particular, those of straight lines, such as the edge of a sheet of paper), it is easy to see that the profile has one minimum and has a relatively flat bottom, ie. the correct answer is "Option D" (full marks are given also for Option B). This profile implies that water is pushed away from the magnet, which means $\mu < 1$ (recall that ferromagnets with $\mu > 1$ are pulled).

Part B. Exact shape of the water surface *(7 points)*

i. (1.6 pts) The height of the spot on the screen y is tabulated below as a function of the horizontal position x of the caliper. Note that the values of y in millimetres can be rounded to integers (this series of measurements aimed as high as possible precision).

x (mm)	10	15	20	25	30	32	34	36
y (mm)	11.5	15.6	19.8	24.3	30.2	33.2	37.2	40.5
x (mm)	38	40	42	44	46	48	50	52
y (mm)	42.2	41.4	40.3	40.3	40.8	42	43.2	44.4
x (mm)	54	56	58	60	62	64	66	68
y (mm)	45.3	45.8	45.4	44.4	43.6	46.2	50	53.6
x (mm)	70	72	74	76	78	80	85	90
y (mm)	56.7	59.5	61.6	63.5	65.3	67	70.9	74.9

ii. (0.7 pts)



On this graph, the data of to two different water levels are depicted; blue curve corresponds to a water depth of *ca 2 mm* (data given in the table above); the violet one — to 1 mm.

iii. (0.5 pts) If the water surface were flat, the dependence of x on y would be linear, and the tangent of the angle α_0 would be given by $\tan\alpha_0 = \frac{\Delta y}{\Delta x}$, where Δx is a horizontal displacement of the pointer, and Δy — the respective displacement of the spot height. For the extreme positions of the pointer, the beam hits the water surface so far from the magnet that there, the surface is essentially unperturbed; connecting the respective points on the graph, we obtain a line corresponding to a flat water surface — the red line. Using these two extreme data points we can also easily calculate the angle $\alpha_0 = \arctan\frac{74.9-11.5}{90-10} \approx 38°$.

iv. (1.4 pts) For faster calculations, $y - y_0 - (x - x_0)\tan\alpha_0$ (appearing in the formula given) can be read from the previous graph as the distance between red and blue line; the red line is given by equation $y_r = y_0 + (x - x_0)\tan\alpha_0$. One can also precalculate $\frac{1}{2}\cos^2\alpha_0 \approx 0.31$. The calculations lead to the following table (with $z = \tan\beta \cdot 10^5$; as mentioned above, during the competition, lesser precision with two significant numbers is sufficient).

x (mm)	10	15	20	25	30	32	34	36
z	0	10	27	66	204	303	473	591
x (mm)	38	40	42	44	46	48	50	52
z	597	428	239	128	53	26	0	-26
x (mm)	54	56	58	60	62	64	66	68
z	-72	-145	-278	-449	-606	-536	-388	-254
x (mm)	70	72	74	76	78	80	85	90
z	-154	-74	-40	-20	-6	2	-2	0

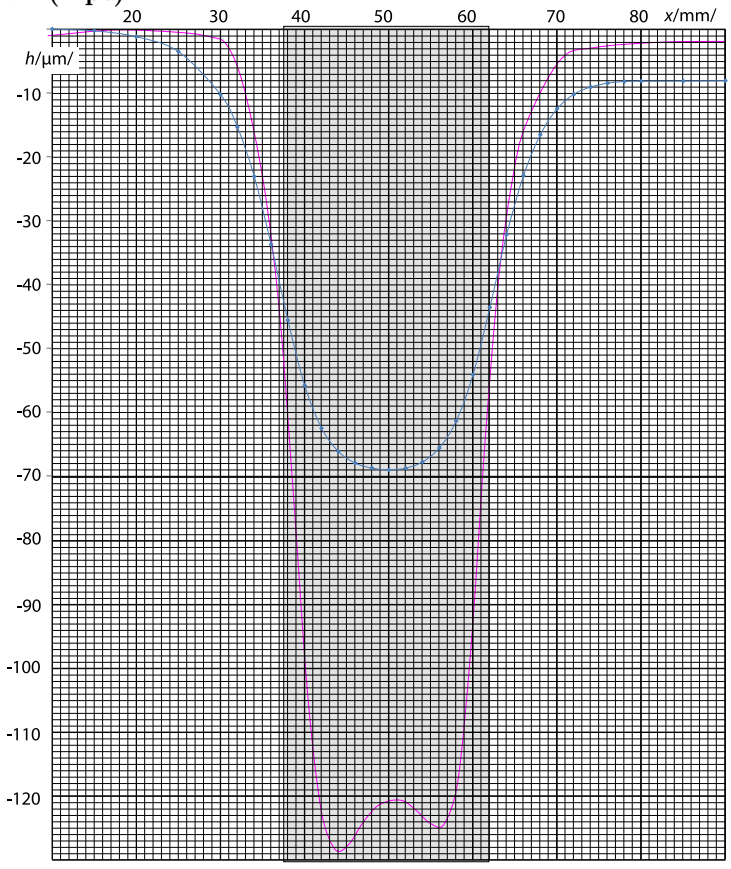
v. (1.6 pts) The water height can be obtained as the integral $h = \int \tan\beta dx$. Thus, we calculate the water height row-by-row, by adding to the height in the previous row the product of the horizontal displacement $x_{i+1} - x_i$ with the average slope $\frac{1}{2}(\tan\beta_{i+1} + \tan\beta_i)$.

x (mm)	10	15	20	25	30	32	34	36
$-h$ (μm)	0	0	1	4	10	15	23	34
x (mm)	38	40	42	44	46	48	50	52
$-h$ (μm)	46	56	63	66	68	69	69	69
x (mm)	54	56	58	60	62	64	66	68
$-h$ (μm)	68	66	61	54	44	32	23	17
x (mm)	70	72	74	76	78	80	85	90
$-h$ (μm)	12	10	9	8	8	8	8	8

Note that the water level height at the end of the table should be also 0 (this corresponds also to an unperturbed region); the non-zero result is explained by the measurement uncertainties. One can improve the result by subtracting from h a linear trend $8\,\mu\text{m} \cdot \frac{x-10\,\text{mm}}{80\,\text{mm}}$.

If the water level above the magnet is 1 mm, the water level descends below its unperturbed level at the axis of the magnet by ca 120 μm.

vi. (1 pt)



Similarly to the previous figure, blue curve corresponds to a water depth of *ca* 2 mm, (data given in the table above), and the violet one — to 1 mm.

The position of the magnet can be found by measuring the caliper (find the positions when the laser beam hits the edges of the magnet and determine the distance between these positions — the result is ca 24 mm), and using the symmetry: magnet is placed symmetrically with respect to the surface elevation curve.

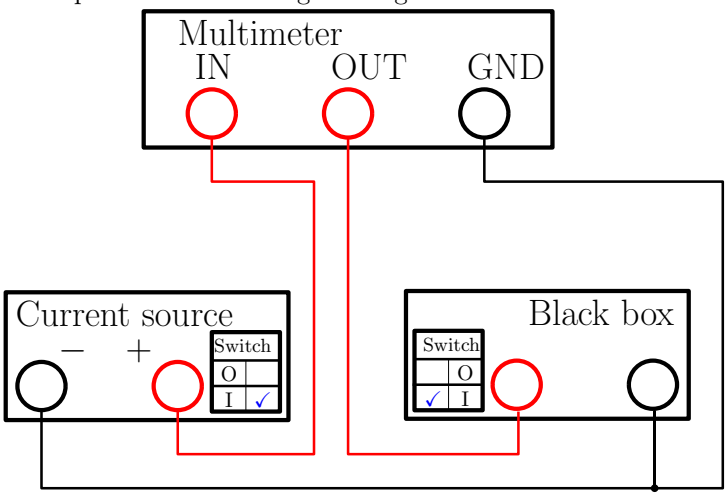
Part C. Magnetic permeability *(2 points)*

Water surface takes an equipotential shape; for a unit volume of water, the potential energy associated with the magnetic interaction is $\frac{B^2}{2\mu_0}(\mu^{-1}-1) \approx B^2\frac{1-\mu}{2\mu_0}$; the potential energy associated with the Earth's gravity is $\rho g h$. At the water surface, the sum of those two needs to be constant; for a point at unperturbed surface, this expression equals to zero, so $B^2\frac{\mu-1}{2\mu_0} + \rho g h = 0$ and hence, $\mu - 1 = 2\mu_0\rho g h/B^2$. Here, $h = 120\,\mu$m stands for the depth of the water surface at the axis of the magnet; note that we have compensated the cumulative error as described at the end of the previous task and obtained h as the difference between the depth at the magnet's axis (121 μm) and the half-depth at the right-hand-side of the graph (1 μm). Putting in the numbers, we obtain $\mu - 1 = -1.2 \times 10^{-5}$.

Problem E2. Nonlinear Black Box *(10 points)*

Part A. Circuit without inductance *(7 points)*

It is possible to make all the measurements needed for this problem with a single circuit as shown in the figure. While the current source is switched on, we are charging the capacitor in the black box, until the current $I(V_{\max})$ through the nonlinear element equals to the output current I_0 of the current source. $V_{\max} = 540 \pm 40\,\text{mVs}$ varies from one experimental setup to another. When the current source is switched off or disconnected, the capacitor will discharge through the nonlinear element.



i. (1 pt) During charging of the capacitor from $V = 0$ to $V = V_{\max}$ we note that the output of the current source is constant ($I_0 = 6.0\,\text{mA}$) close to the precision of the multimeter.

ii. (1.2 pts) Using the definition of differential capacitance, we can calculate the current through the capacitor in the black box from the time derivative of the voltage on the black box.

$$I_c = \frac{dQ}{dt} = \frac{dQ}{dV}\frac{dV}{dt} = C(V)\dot{V}$$

There are several ways to determine the capacitance used in the black box based on chosen voltage.

- When the voltage on the black box is close to zero, the current through the nonlinear element is also close to zero, because $I(V = 0) = 0$. After switching the current source on, most of the input current I_0 will at first go through the capacitor.

  $$C_0 = I_0/\dot{V}_{\uparrow}(V = 0)$$

  This can be measured more precisely after first reversing the polarity of the current source and charging the capacitor backwards, because the multimeter does not display derivatives when they change sharply (as in few moments after switching the current source on).

  Example measurements taken this way follow.

$V_{\uparrow}(0)$ (mV/s)	3.51	3.32	3.55
C_0 (F)	1.71	1.81	1.69

  $$C_0 = 1.74\,\text{F}$$

- When the voltage on the black box is $V_{\max}$, the current through the nonlinear element is I_0. Switching the current source off, we will have the capacitor discharging with the same current.

  $$C_0 = -I_0/\dot{V}_{\downarrow}(V = V_{\max})$$

- We can also measure the capacitance for any intermediate voltage as in **A-iv**.

iii. (2.2 pts) If we neglect the nonlinearity of the capacitor, there are (at least) two ways to obtain the current–voltage characteristic of the nonlinear element in the black box.

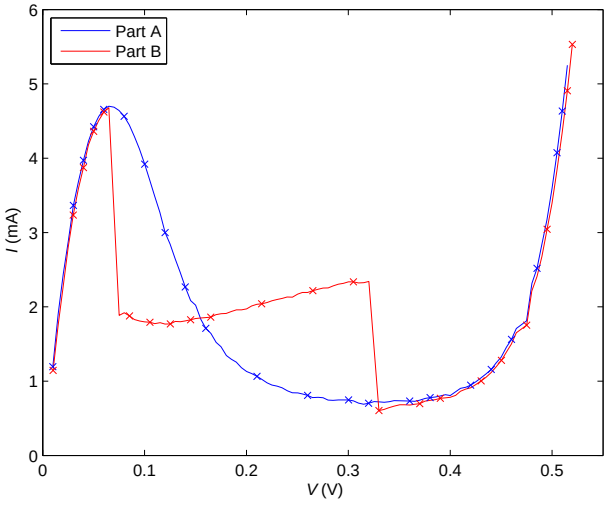
- Applying Kirchhoff's I law to the charging capacitor,

  $$I(V) = I_c - C_0\dot{V}_{\uparrow}(V).$$

  An $I(V)$ characteristic obtained by charging the capacitor is shown on the following figure.

- Applying Kirchhoff I law to the discharging capacitor,

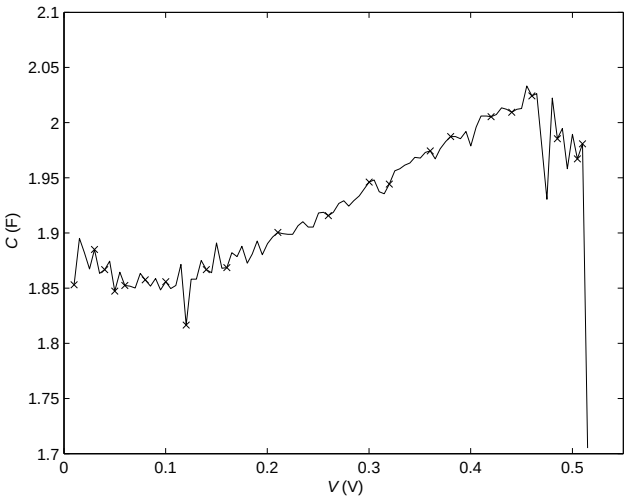
  $$I(V) = -C_0\dot{V}_{\downarrow}(V).$$



iv. (2.6 pts) In order to obtain the differential capacitance, we solve a system of linear equations by eliminating $I(V)$:

$$\begin{cases} I_0 = \dot{V}_{\uparrow}C(V) + I(V) \\ I(V) = -\dot{V}_{\downarrow}C(V); \end{cases} \quad \Longrightarrow \quad C(V) = \frac{I_0}{\dot{V}_{\uparrow} - \dot{V}_{\downarrow}}.$$

Therefore we need to take measurements during both charging and discharging the capacitor in the black box at the same voltages. A graph of measurement results follows.



Part B. Circuit with inductance *(3 points)*

Measuring and plotting the current–voltage characteristic of the nonlinear element in the same way as in part **A-iii**, we obtain a graph that differs only in the negative differential resistance ($I'(V) < 0$) region, in our case $70\,\mathrm{mV} < V < 330\,\mathrm{mV}$. This is the region where, when we look at small-signal oscillations, the nonlinear element behaves as a negative-valued Ohmic resistance. After enabling the inductance we have a LC circuit whose oscillations are amplified (instead of being dampened) by the negative differential resistance. Because the resonant frequency $\omega = \sqrt{\frac{1}{LC_p}} \sim 30\,\mathrm{MHz}$ (with C_p being the capacitance of the nonlinear element) is high, we actually measure the average current through the nonlinear element, while the real current oscillates all over the region of negative differential resistance.