超流体中的涡旋

简介

超流动性是没有摩擦的流动。日常经验告诉我们一种普通流体 (比如室温下的水) 的运动总是伴随着能量的粘性耗散以致于流动逐渐变慢，除非通过外力来保持。与此相反，超流体没有动能损失：一旦被激发，超流体的运动可以无限地继续下去。超流动性最初是在液氦实验中被发现的。

我们在零度下研究超流体氦的特性。这里超流体氦被当作不可压缩的液体，其密度记为 ρ. 流动的连续性（对于一个给定的无限小体积空间，流进和流出的质量是相等的）意味着通过一个封闭表面的氦的速度 $\vec{v}$ 的通量为零。从这一点来说，超流体的速度和磁场强度相似。与磁力线类似，速度与“流线”(”streamlines”) 的切线方向相同，流线密度正比于速度的大小。

真正的超流体有一个重要的性质，无旋性，即超流体的速度沿液氦中任意闭合回路积分结果为零.

$$\int_L \vec{v} \cdot d\vec{l} = 0 \tag{1}$$

若沿着一个细“涡旋丝”(”vortex filament”) 不存在超流性，则上述结论需要修改。丝的厚度大约是原子大小，a，但涡旋会在周围的超流体中产生长程速度场：速度绕涡旋丝的环路积分是旋度量子（circulation quantum）[1]

$$\left| \int_L \vec{v} \cdot d\vec{l} \right| = 2\pi\kappa, \tag{2}$$

若速度环路积分的积分路径在不穿过涡旋丝的情况下可以被收缩至一个点，则环路积分等于零 (见图 1)。基于此，我们可以将超流体中涡旋丝产生的速度场和相同形状电流产生的磁场作类比：两个速度分布可以矢量叠加，其结果和对应形状单位电流产生的磁场相同（只差一个带量纲的因子）。

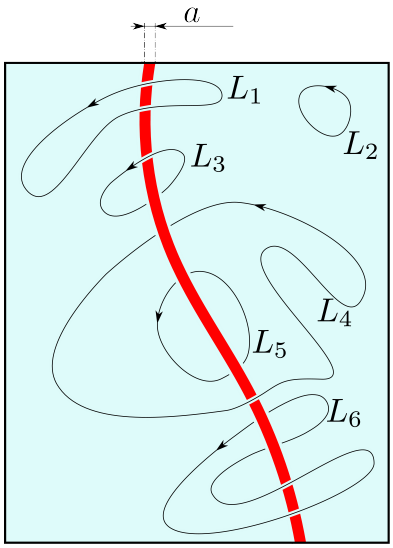


图 1：超流体 (浅蓝色) 中的涡旋丝 (红色)。速度沿路径 L_1, L_2, L_5, 和 L_6 的旋度为零，但沿 L_3 和 L_4 的旋度等于 $\pm 2\pi\kappa$. 注意沿 L_3 和 L_4 的旋度方向相反。

[1]旋度量子化是一种宏观量子效应，对应于玻尔模型中角动量的量子化。旋度量子可以表示成 $\kappa = h/m_{\mathrm{He}}$, 其中 m_{He} 表示氦原子的质量。

Part A 稳态涡旋丝（0.75 分）

考虑一个圆柱形烧杯 (半径 $R_0 \gg a$) 中的超流体氦，及位于其中心的一个竖直的涡旋丝，见图 2.

A.1	画出流线。求出在点 $\vec{r}$ 处速度 v 的表达式。	0.25pt

A.2	推导出涡旋附近自由表面的形状 (表面高度表示成坐标的函数 $z(\vec{r})$). 重力加速度为 g ，忽略表面张力。	0.5pt

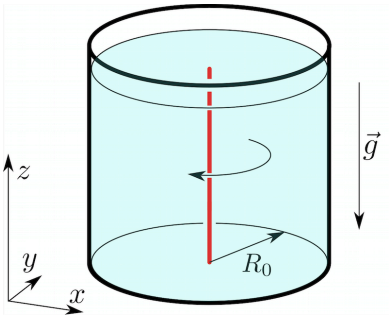


图 2：沿烧杯中心轴线的直涡旋 (straight votex)。

Part B. 涡旋运动（1.4 分）

自由的涡旋会随着流动在空间中运动[2]. 也就是说丝的每一个小单元都随着流体在该单元位置处的速度 $\vec{v}$ 运动。

例如，考虑两个旋转方向相反的直涡旋并排放置，相互间距离为 r_0，见图 3。每个涡旋在另一个涡旋的轴上产生速度 $v_0 = \kappa/r_0$. 因此涡旋对 (vortex pair) 以恒定速度 $v_0 = \kappa/r_0$ 直线运动，并且二者之间的距离保持不变。

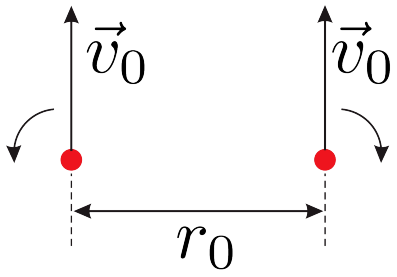


图 3：两个平行放置且旋度相反的涡旋丝

B.1	考虑两个相同的直涡旋并排放置，相互距离为 r_0 ，如图 4 所示。求出涡旋初始速度的表达式，并且画出它们的轨迹图 (trajectories) 。	0.25pt

[2]这是动量守恒的结果，见下一节。

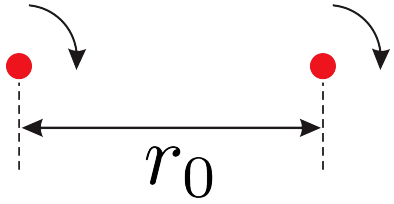


图 4：两个平行放置且旋度相同的涡旋丝

圆柱形烧杯中的氦 (见 Part A) 充满了三角形状栅格 ($u \ll R_0$) 分布的相同竖直涡旋，见图 5。

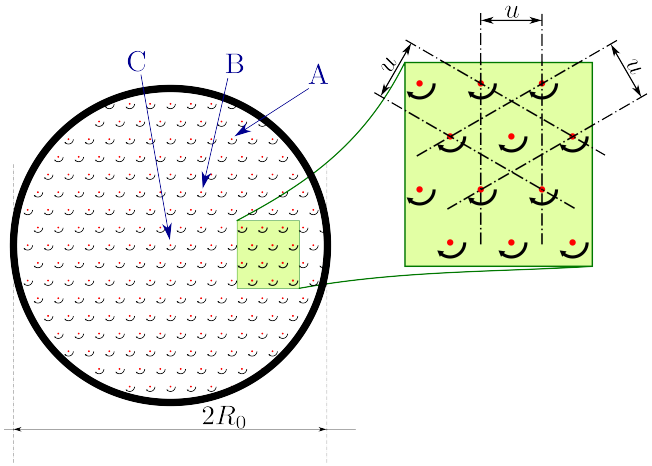


图 5：烧杯中的三角形状栅格分布的涡旋。此为俯视图。

B.2	画出涡旋 A, B 和 C (位于中心) 的轨迹。	0.15pt
B.3	推导出位于位置 $\vec{r}$ 处的一个涡旋的速度 $v(\vec{r})$ 的表达式。	0.4pt
B.4	推导给出涡旋 A 和 B 在时间 t 时的距离 AB(t). 把 AB(0) 当作已知。	0.35pt
B.5	推导给出平滑 (忽略栅格结构) 的自由氦表面形状 (free helium surface shape) $z(\vec{r})$.	0.25pt

Part C. 动量和能量（1.75 分）

涡旋系统的能量主要来自于长程速度场的贡献。这种速度场对丝的具体结构不敏感。丝本身不能用宏观理论来合理描述，并且这些明显的奇异位置 (无穷大) 无重要意义。绕着涡旋丝、半径为 a 的细管内的实际物理量，比如能量，应该被忽略。细管外超流动能密度 $\rho v^2/2$ (这里 $\rho = \text{const}$) 和磁场能量密度 $B^2/(2\mu_0)$ 相似——它们都是各自变量的二次函数。这种相似性及电流产生的磁场和涡旋（与电流形状相同）产生的速度场之间的类比，有助于我们计算一个给定系统的能流。例如，给定一圆形线圈的电感 $L \approx \mu_0 R \log(R/a)$ ，其中 R 是线圈半径，a

是电线的半径。我们得到超流体涡旋环路的能量[3]

$$U \approx 2R\rho\pi^2\kappa^2 \log(R/a) \tag{3}$$

总的流体动量同样取决于长程速度分布，并可以通过对动量密度 $\rho\vec{v}$ 作积分得到。同样，考虑一个由位于 xy 平面的圆形涡旋回路产生的流动。很明显，基于对称性考虑，总的动量只有 z 分量：

$$P = \int \rho v_z dV = \rho \int\int \underbrace{\left(\int v_z dz\right)}_{q(x,y)} dxdy \tag{4}$$

最里面的积分事实上是沿某些平行于 z 轴的路径的积分, 见图 6. 从旋度恒等式 (公式 2) 可知

$$q(x,y) = \int_{L(x,y)} \vec{v} \cdot d\vec{l} \tag{5}$$

是分段常数。当路径从圆环外面穿过，上式为 0；当路径从圆环里面穿过，上式为 $2\pi\kappa$. 总的动量因此可表示为

$$P = \rho \cdot \pi R^2 \cdot 2\pi\kappa = 2\pi^2\rho R^2\kappa \tag{6}$$

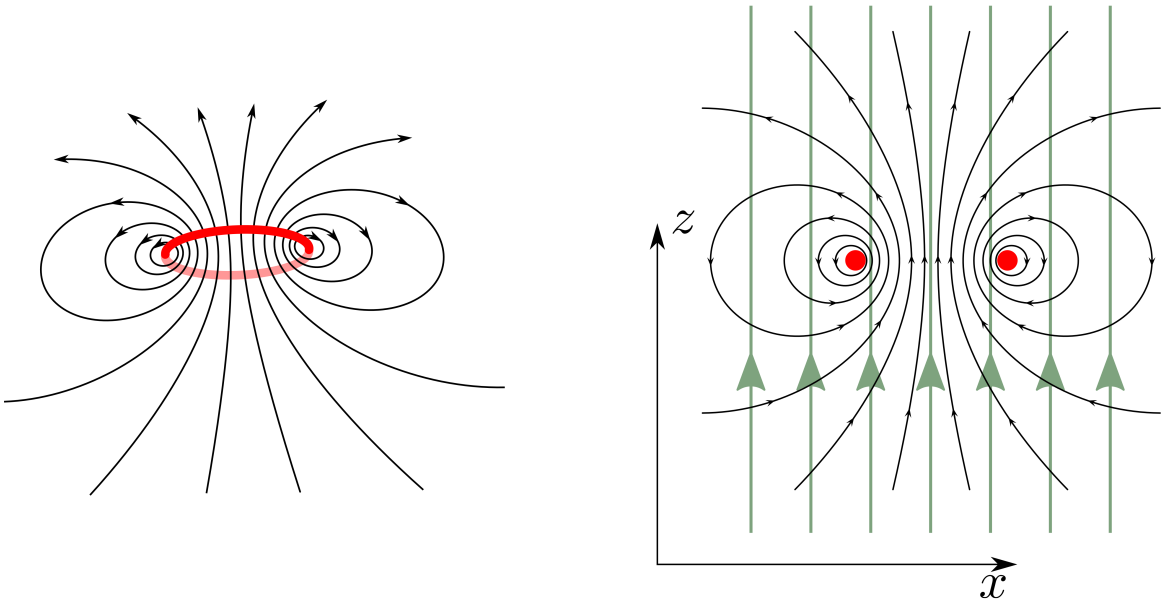


图 6：圆形涡旋环路的速度场和用于计算 $q(x,y)$ 的积分路径 (绿色)

[3]仅当 $\log R/a \gg 1$，这种表达有效。

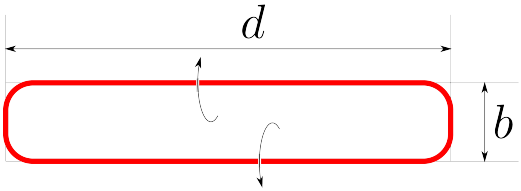


图 7：一个近似为矩形的涡旋环路，$b \ll d$

C.1	考虑一个近似为矩形的涡旋线圈 $b \times d, b \ll d$, 见图 7. 标出其动量 $\vec{P}$ 的方向。推导出动量大小的表达式。	0.3pt
C.2	计算其能量 U.	0.7pt
C.3	假设我们在 x 方向上平移一个长直涡旋丝，平移距离为 b , 见图 8. 流体动量改变多少？标出动量改变方向。丝的长度 (受容器壁的限制) 是 d .	0.75pt

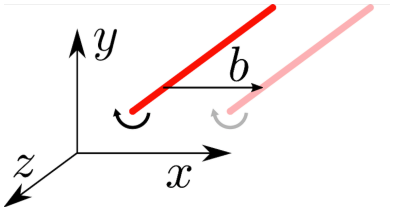


图 8：当涡旋相对于流体平移时会有动量改变

Part D. 被捕获的电荷（2.85 分）

如果把电子注入氦中，电子会被涡旋丝捕获。这里和下面的部分中，忽略氦的极化率 ($\epsilon = 1$).

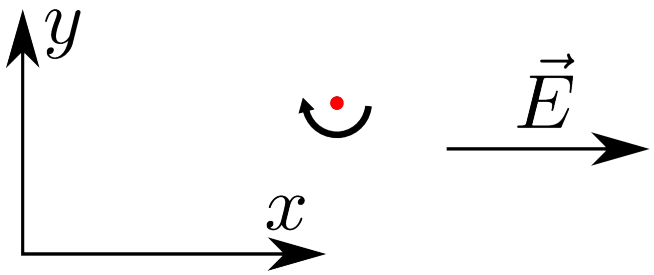


图 9：均匀电场中的直涡旋

D.1	考虑均匀电场 $\vec{E}$ 中具有均匀线电荷密度 $\lambda < 0$ 的直涡旋。画出涡旋轨迹图。推导给出其速度随时间的变化。	0.5pt

在一个均匀电场 $\vec{E}$ 中放置一个半径为 R_0 ，具有均匀线电荷密度 $\lambda < 0$ 的圆形涡旋环路，电场方向垂直于环路所在的平面，并与环路动量 $\vec{P}_0$ 方向相反。

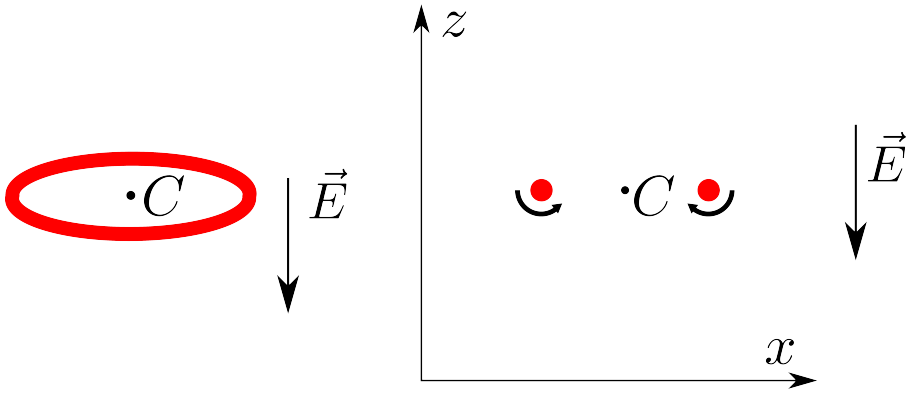


图 10：（左）均匀电场中的涡旋环。（右）环的截面。

D.2	画出环中心 C 的轨迹。求出环半径随时间的变化函数。	0.6pt
D.3	推导给出其速度 $v(t)$ 随时间的变化。	1.5pt
D.4	在时刻 t^* 撤去电场，此时速度为 $v^* = v(t^*)$. 推导给出在时刻 $t > t^*$ 时环的速度 $v(t)$.	0.25pt

Part E. 边界的影响（3.25 分）

固体器壁会改变涡旋丝的速度场，这是因为流体不能从器壁中流过。在数学上，这要求在器壁表面，垂直于器壁表面的速度分量为零。

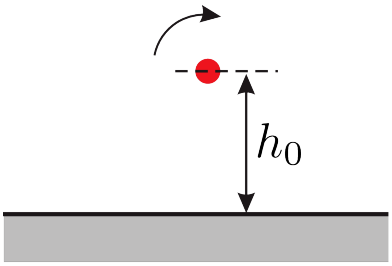


图 11：一个平面器壁附近的直涡旋丝。

E.1	初始时，一个直涡旋丝与平面器壁相距 h_0，画出其后涡旋丝的运动轨迹，求出速度作为时间的函数。	0.5pt

考虑一个放置于直角器壁附近的直涡旋丝，其至两壁距离都为 h_0.

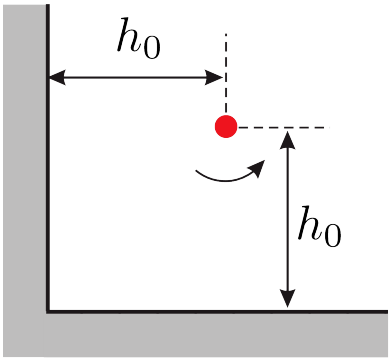


图 12：放置在直角器壁附近的直涡旋丝。

E.2	涡旋的初始速度 v_0 是多少?	0.75pt
E.3	画出涡旋的运动轨迹。	0.5pt
E.4	经过很长时间后，涡旋的速度 v_∞ 是多少?	1.5pt

超大质量双黑洞系统（Supermassive Black Holes Binary ）的演化

前言

引力波是爱因斯坦广义相对论的重要预言之一。引力波是以光速传播的时空波，与电磁波类似。直接探测引力波非常困难，然而，首个引力波讯号于 2015 年 9 月 14 日由激光干涉引力波天文台 (LIGO) 及处女座干涉仪 (VIRGO) 联合观测发现。

大质量物体的迅速运动会辐射引力波。两个超大质量黑洞 (Supermassive Black Holes (简称 SBH)) 合并是最强大的引力波波源。根据广义相对论的预测，黑洞是极端高密度物体，其质量可能非常大。在本题解答中不再需要黑洞其它特有的性质。

在普遍接受的星系演化理论中，假设星系中心存在超大质量黑洞 (SBH)，其质量约为 $10^5 - 10^9$ 倍太阳质量。星系是巨大的恒星系统，拥有 $10^{10} - 10^{11}$ 颗恒星。在演化过程中，两个星系会碰撞并合并为一。起初分别位于两个星系中心的两个超大质量黑洞 (SBH) 会怎样演化呢？超大质量双黑洞系统（the SBH binary）的演化可分为三个主要阶段。在每一阶段，尽管彼此表现的物理现象不同，但是两个超大质量黑洞 (SBH) 都会互相靠近。在本题的前三部分，我们将会分别处理这些现象。在第四部分，我们将会利用所获得的关系，以计算超大质量双黑洞系统（SBH binary）演化的总时间。

当演化结束，两个超大质量黑洞 (SBH) 会最终互相靠近，并合并为一个黑洞。合并过程持续约一小时，其间会伴随爆发强烈的引力辐射。未来的观察如激光干涉太空天线 (LISA) 能够探测这些引力辐射。伴随着引力波天文学的曙光，有关超大质量黑洞 (SBH) 系统演化的研究仍然在进行中。

注意事项

1. 所以有关距离数值答案以秒差距 (pc) 表示，而时间间隔以 G 年 (Gy) 表示。我们以太阳质量 (M_S) 作为参考质量。你可能需要以下数值：

$1\ \mathrm{pc} = 3.1 \times 10^{16}\ \mathrm{m}$,

$M_s = 2.0 \times 10^{30}\ \mathrm{kg}$,

$t_H = 13.7\ \mathrm{Gy}$, 宇宙年龄，13.7×10^9 年,

$G = 6.67 \times 10^{-11}\ \mathrm{N} \times \mathrm{m}^2/\mathrm{kg}^2$,

$c = 3.0 \times 10^8\ \mathrm{m/s}$.

2. 当你见到“**估算 (estimate)**”这个词语时，你的答案不需要有准确值，但与真值的差异少于 10 倍。相反，当你见到“**求出 (find)**”这个词语时，你必须给予准确结果。“**计算 (calculate)**”这个词语表示你的答案必须以数值表示。

3. 在整个问题中，均假设星系中每颗恒星的质量等于太阳质量 $m = M_S$。

4. 在整个问题中，除了引力波，我们不考虑广义相对论的影响。所有恒星及黑洞均以牛顿万有引力定律中的质点处理。

Part A. 动力学摩擦 (1.6 分)

在这部分，我们将研究星系的简化模型。你可以忽略星系中恒星的速度及假设有恒定的恒星数密度 n. 星系的特征大小是 R . 由于数密度足够小，故恒星的碰撞非常罕见，可以忽略不计。我们先考虑一个超大质量黑洞（SBH），其质量 $M \gg m$，并以速度 v 在星系中运动。令人惊讶的是，该黑洞（SBH）受到来自恒星的平均作用力并非等于零。这个力减慢了 SBH 的运动，因此称为动力学摩擦力。这部分将致力于研究这个力。

A.1 我们以 SBH 为参考系，分析一个恒星以瞄准距离（impact parameter）b 经过的过程 (图 1)。假设 0.75pt

$$b \gg b_1 = \frac{GM}{v^2}. \tag{1}$$

恒星的偏转角 $\alpha = kb_1/b$，k 为系数。准确计算 k 的数值。如若你未能准确计算 k，在余下部分假设 $k = 1$。

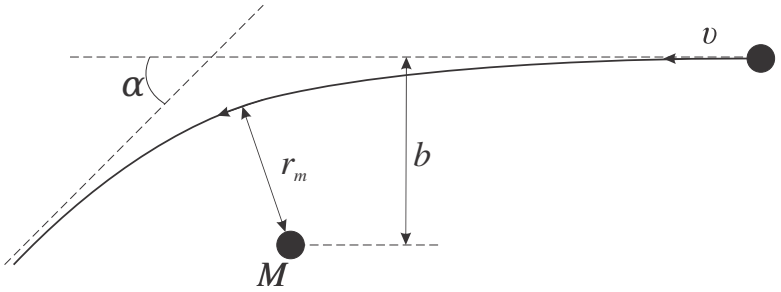


图 1：一颗恒星在一个质量为 M 的 SBH 作用下发生偏转。瞄准距离（impact parameter）是 b, 恒星和物体之间的最短距离是 r_m .

A.2 沿 SBH 的速度方向建立 Ox 轴。求出从恒星转移到 SBH 的动量分量 Δp_x. 0.25pt

A.3 估算 SBH 受到的平均作用力 F_{DF} ，按瞄准距离 b 来求平均. 忽略 $b < b_1$ 的那些恒星的贡献. 假设 SBH 处于星系的中心. 用 M, v, R, G 和星体密度 $\rho = mn$ 来表示 F_{DF}. 0.4pt

A.4 如你在之前任务中得到的一样，F_{DF} 的表达式包含因子 $\log R/b_1$, 此后我们将此记为 $\log\Lambda$. 当 $M = 10^8\ M_S, R = 20\ \text{kpc} = 20 \times 10^3\ pc$, 速度 $v = 200\ \text{km/s}$ 时，计算 $\log\Lambda$. 0.2pt

Part B. 引力弹弓 (3.0 分)

在这一部分中，我们考虑一个位于星系中心的系统，该系统由两个具有相同质量 $M \gg m$ 的超大质量黑洞 (SBH) 所组成，我们称之为**超大质量双黑洞系统（SBH binary）**。我们假定这个超大质量双黑洞系统附近没有其它恒星。每一个超大质量黑洞在另一个超大质量黑洞的引力场中都有一个半径为 a 的圆形轨道。

B.1	求出每个超大质量黑洞的轨道速度 v_{bin} . 求出超大质量双黑洞系统的总能量 E ，用 a, G 和 M 来表达。	0.25pt

在与该系统距离远大于 a 的位置有很多恒星。在整个星系的引力场作用下恒星沿着不同的复杂的轨道运动。恒星的运动可看作是混沌的，就像理想气体中分子的运动一样。我们假定恒星运动速度相同，且大小 $\sigma \ll v_{bin}$ ，平均质量密度记为 ρ . 在这种情况下，动力学摩擦不再影响超大质量双黑洞系统，能量损失由其它因素造成。

B.2	让我们来解决一个与此有关的问题。让一个质量为 m 的恒星经过一个位置不动、质量为 $M_2 \gg m$ 的质点附近。在运动过程中恒星和质点之间的最小距离记为 r_m. 恒星在较大距离时的速度记为 σ. 求出瞄准距离 b 的准确值。	0.5pt

若一颗恒星趋近一个超大质量双黑洞系统至约 a 的距离，则会和双星发生复杂的三体相互作用，其结果几乎总是射出一个速度约为 v_{bin} 的恒星（这里的速度是相互作用后在一个很远距离处的速度）。我们将这种强的相互作用称为一颗恒星与一个超大质量双黑洞系统的碰撞。碰撞后的恒星的加速和出射被称为“**引力弹弓 (gravitational slingshot)**”

B.3	估算超大质量双黑洞系统和恒星两次连续碰撞之间的特征时间间隔 Δt . 请注意 $\sigma \ll v_{bin}$.	1.0pt

B.4	估算超大质量双黑洞系统的能量损失率 dE/dt. 估算半径的变化率 da/dt. 用参量 a, ρ, σ, G 表示结果.	0.25pt

B.5	让我们指定系统初始半径为 a_1. 估算由于“引力弹弓”效应，半径减小一半所用的时间 T_{SS}. 当 $\sigma = 200\,\mathrm{km/s}, a_1 = 1\,\mathrm{pc}, \rho = 10^4 M_S/\mathrm{pc}^3$ 时，计算 T_{SS}.	1.0pt

Part C. 引力波辐射 (1 分)

在这部份我们将研究一个与其他恒星没有任何相互作用，且拥有两个相同质量黑洞的超大质量双黑洞系统 (SBH Binary)。此系统中，能量的流失是由于引力波对外的辐射。其能量的流失率是：

$$\frac{dE}{dt} = -\frac{1024}{5}\frac{G}{c^5}(\omega^3 I)^2, \tag{2}$$

其中 w 是超大质量双黑洞系统的角速度和 $I = 2Ma^2$ 是系统的四极矩。

C.1	求出因辐射引力波而导致的超大质量双黑洞系统的半径随时间的变化率 da/dt.	0.2pt

当该超大质量双黑洞系统的轨道半径 a 接近于其内黑洞的引力半径时：

$$r_g = \frac{2GM}{c^2}, \tag{3}$$

两个超大质量黑洞便会即时进行合并。

C.2	假设该系统的初始半径 $a_2 >> r_g$. 请估算该系统因辐射引力波而导致半径收缩至 r_g 所需的时间 T_{GW}. T_{GW} 表示成 a_2, M, c 和 G 的函数。	0.7pt

C.3	假设超大质量双黑洞系统需宇宙年龄的时间：$T_{GW} = t_H$ 才能进行合并，请计算一个拥有两个质量 $M = 10^8 M_S$ 黑洞的超大质量双黑洞系统的初始半径 a_H.	0.1pt

Part D. 整个演化的过程 (4.4 分)

在这部分，我们将利用以上各部分获得的结果，并考虑天体物理的真实情况。考虑两个中心各有一个质量 $M = 10^8 M_S$ 的 SBH 的星系合并成一个新星系。合并后的新星系呈球状对称且半径为 $R = 20kpc = 20 \times 10^3 pc$. 假设这个星系的密度随至星系中心距离 r 的变化为

$$\rho(r) = \frac{\sigma^2}{4\pi G r^2}, \tag{4}$$

其中 $\sigma = 200kms^{-1}$.

D.1	若有一物体在该星系的引力作用下在半径 $a < R$ 的圆形轨道上运动。在不用考虑动力学摩擦的情况下，求出该物体的速度 v.	0.25pt

当两个星系刚合并成一个新星系后，其中的两个超大质量黑洞可在该星系内任意位置，并互不影响。我们将考虑其中一个超大质量黑洞。假设它绕星系中心作半径 $a < R$ 的圆周运动，并且由于动力学摩擦缓慢损失能量。

D.2	请估算轨道半径随时间的改变率 da/dt. 在 Part A，我们忽略了星系中恒星的速度，现实情况下星系内各恒星速度不为零，也不都有相同速度 σ，但其量级与 σ 相同，而 SBH 相对其它恒星的运动速度也为同一量级，所以你仍然能够使用 A.3 获得的公式来作估算。你需要用到公式 (4) 中的 $\rho(r)$，并假设 $log\Lambda$ 是 A.4 中计算得到的常数。	0.75pt

经过一定时间后，两个超大质量黑洞将接近星系的中心。假设这两个超大质量黑洞在星系的引力下均绕星系中心作半径为 a 的圆周运动。

D.3	估算这两个超大质量黑洞到达其间相互引力不能被忽视时的临界半径 a_1，并计算出数值. 当到达这个临界半径时，便认为超大质量双黑洞系统已形成 (图 2).	0.3pt

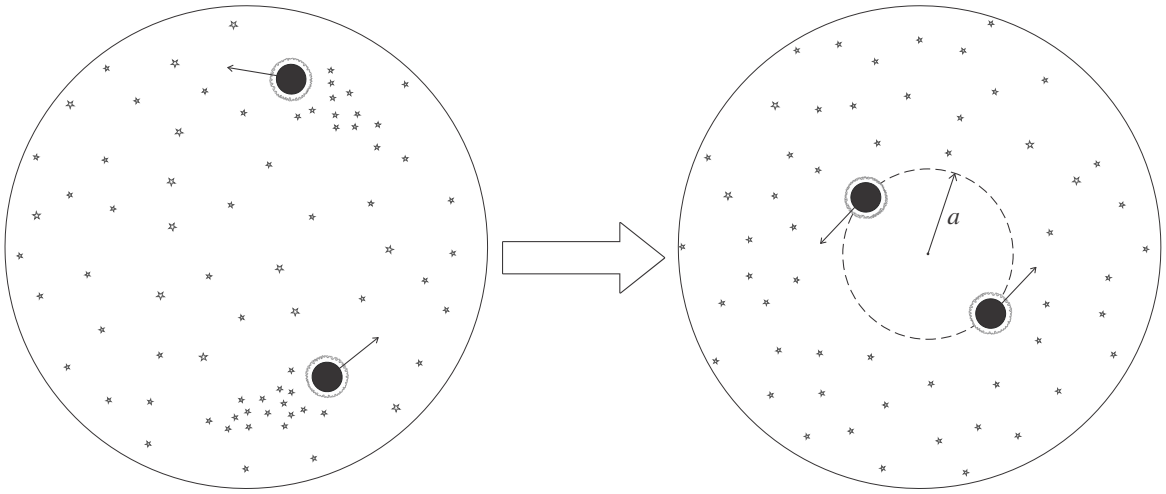


图 2: 超大质量双黑洞系统形成前后的演化。

D.4	我们假设，两个超大质量黑洞星系在距星系中心距离为 $a_0 = 2\,\mathrm{kpc} = 2 \times 10^3\,\mathrm{pc}$ 处合并之后，计算由于动力学摩擦，两个超大质量黑洞形成双星所需要的时间 T_1.	0.75pt

形成双星之后，两个超大质量黑洞从星系的中心射出其所有恒星，并独自留在原处。由于这时的动力学摩擦变得无效，超大质量黑洞由于弹弓效应开始失去能量。你可以假设恒星绕双星运动的速度是 σ，根据公式（4），星体密度为 $\rho_1 = \rho(a_1)$. 弹弓效应迅速减少体系的半径，经过一段时间之后，体系主要由于引力波辐射而开始失去能量。

D.5	当双黑洞的半径 $a < a_2$ 时，引力波辐射导致能量损失。	0.3pt

D.6	估算双黑洞的半径从 a_1 减小到 a_2 （弹弓阶段）所需要的时间 T_2 。估算双黑洞的半径从 a_2 减小到几乎为 0（引力波辐射阶段）所需要的时间 T_3 .	1.75pt

D.7	用上述给的参数，计算两个超大质量黑洞从星系合并到超大质量黑洞合并之间演化所需要的总时间 T_{ev}.	0.3pt

历史评述。长期以来，天体物理学家一直认为超大质量双黑洞在弹弓阶段就停止了演化，这是由于双黑洞将碰撞距离较小的、有可能和双黑洞发生碰撞的所有恒星全部射出。这似乎意味着两个超大质量黑洞绝不会发生合并。这个事实被称为**最后的秒差距问题**。

真实的星系具有复杂的非对称性。几年前，人们发现，在复杂形状的星系中，碰撞距离较小的恒星不断被发现。超大质量双黑洞持续失去能量，但比我们这里估计的要慢。最终秒差距问题被成功解决。

太空碎片

简介

在半个多世纪的太空探索中，很多人为造成的物体聚集在地球附近。这些没有特定的目的的物体称为太空碎片。人们通常最关注其中较大的物体，例如那些不能正常工作的卫星和用过的多级火箭最后两级（upper stage），它们在完成其使命后继续处在轨道上。这些较大的太空垃圾彼此碰撞，从而产生成千的碎片，这些碎片使所有目前运行的空间飞行器面临危险。

众所周知的一个假想图像如下：某些太空垃圾相碰可能会产生连锁效应，每一个碰撞导致的新碰撞将产生更多的新碎片，这又增加了新碰撞的可能性。这样的链式反应导致我们失去所有地球附近的人造卫星，使新的空间项目无法进行，这一现象被称为**基斯勒综合症（Kessler syndrome）**。

为了防止上述不希望的结果发生，人们计划组建一个特别行动队，把大的太空垃圾从其现在的轨道拖到地球的大气层或拖到死亡轨道（graveyard orbits）。为此，一个特制的太空装置，即天空拖车将用来俘获太空垃圾。在俘获那些失控的物体之前，我们需要理解其转动动力学机制。

我们建议你参与这一行动的计划，并发现太空垃圾的转动动力学如何在不同因素的影响下发生变化。

火箭残骸（rockets stage）的描述

我们这里考虑的火箭残骸是一个“科博丁 42 号”（Kerbodyne 42）火箭的最后一级的残骸，如图 1. 图中的圆标出了球形燃料箱的边界。

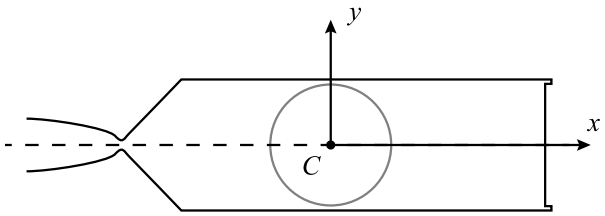


图 1: “科博丁 42 号”（Kerbodyne 42）残骸（upper stage）

我们引入一个随体坐标系 Cxy，坐标原点在质心 C, x 是残骸的对称轴，y 垂直于 x. 绕 x 轴和 y 轴的转动惯量分别为 J_x 和 J_y ($J_x < J_y$).

Part A. 转动 (3.8 分).

考虑残骸的任意一个初始转动，其相应的角动量为 L (图 2), 图中的 θ 为对称轴和角动量之间的夹角。假定此时的燃料箱是空的。且没有任何力和力矩作用在残骸上。

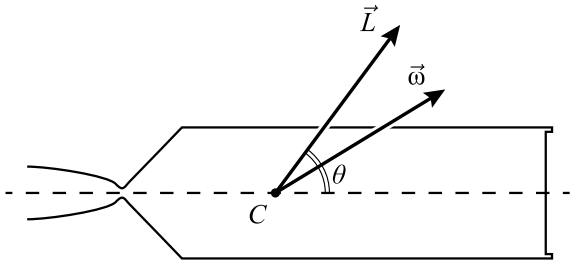


图 2：残骸的转动

A.1 分别求角速度 $\vec{\omega}$ 在 x 轴和 y 轴上的投影。已知 $\vec{L} = J_x\omega_x\vec{e}_x + J_y\omega_y\vec{e}_y$，这里，$x$ 为物体的对称轴，x 和 y 轴上的单位矢量分别为 $\vec{e}_x$ 和 $\vec{e}_y$. 0.2pt

A.2 分别求和转动 ω_x 和 ω_y 相联系的转动能量 E_x 和 E_y . 求残骸的总转动动能 $E = E_x + E_y$，表示为角动量 L 和 $\cos\theta$ 的函数。 0.4pt

在 Part A 的剩余部分中，考虑残骸以初始角动量 L 和 $\theta(0) = \theta_0$ 自由转动。

A.3 用 x_0 表示残骸的对称轴 Cx 初始时相对于惯性参考系的取向。通过守恒定律求角 ψ 的最大值。ψ 为残骸的对称轴 Cx 在自由转动过程中和 x_0 之间的夹角。
注意：由于没有任何外力矩作用在残骸上，角动量矢量为常量。 1.2pt

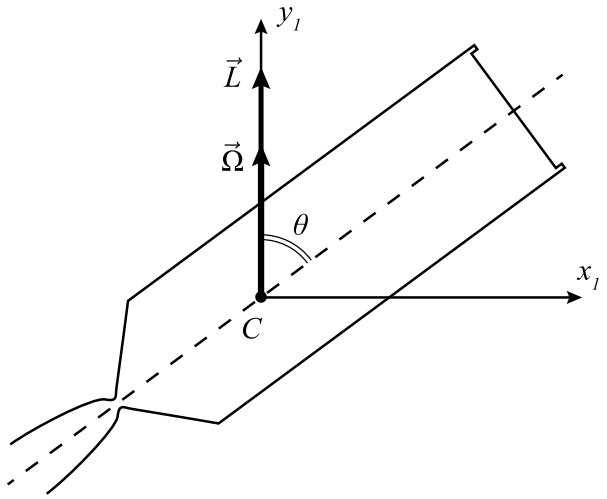


图 3: 进动

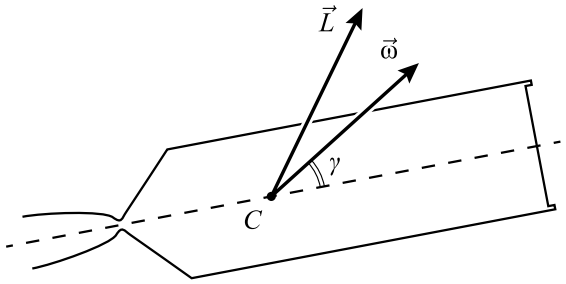
引入参考系 $Cx_1y_1z_1$，其中 y_1 沿恒定角动量 $\vec{L}$ 方向 (图 3). 上述参考系绕 y_1 转动，使残骸对称轴始终位于 Cx_1y_1 平面。

A.4	对给定的 $L, \theta(0)=\theta_0$, 和转动惯量 J_x, J_y ，求参考系 Cx_1y_1 绕 y_1 转动的角速度 $\Omega(t)$ ；求残骸相对于参考系 Cx_1y_1 转动角速度 $\vec{\omega}_s(t)$ 的方向和绝对值，直接表示为时间的函数；$\vec{\omega}_s(t)$ 的方向表示成它与对称轴 Cx 的夹角 $\gamma_s(t)$. 注意：角速度矢量可以相加，即 $\vec{\omega}=\vec{\omega}_x+\vec{\omega}_y=\vec{\Omega}+\vec{\omega}_s$.	2.0pt

Part B. 暂态过程 (Transient Process，1.2 分).

大多数火箭在上升阶段提供动力，当被发射物体和残骸（火箭）分离后，在残骸的燃料箱里还有剩余燃料。假如剩余燃料的质量 m 与残骸的质量 M 相比可以忽略. 燃料箱里的液体燃料晃荡及其粘滞摩擦力将导致能量损失，经过一个非常规的动力学暂态过程后，能量到达极小。

B.1	对任意的初始值 L 和 $\theta(0)=\theta_1\in(0,\pi/2)$ ，经过一个暂态过程后，角度 θ 变为 θ_2，求 θ_2.	0.6pt



图：残骸角速度和对称轴之间的角度

B.2	给定初始角速度 $\omega(0)=\omega_1=1\ rad/s$, 角速度方向与残骸对称轴的夹角 $\gamma(0)=\gamma_1=30^\circ$ ，以及 $J_x=4200\ kg\cdot m^2$ 和 $J_y=15\,000\ kg\cdot m^2$，计算暂态过程后角速度 ω 的值 ω_2 .	0.6pt

Part C. 磁场 (5 分).

地球磁场是影响地球轨道上火箭残骸转动动力学的另一个重要因素。让我们先考虑一个辅助问题。

涡流（Eddy Currents）引起的力矩

将一个厚度为 D、半径为 R 的薄壁非磁性球壳放入一个均匀磁场 $\vec{B}$，该磁场随时间缓慢变化，其时间导数 $\dot{\vec{B}}$ 为常数矢量，与磁场 $\vec{B}$ 方向夹角为 α. 壳材料的电阻率为 ρ.

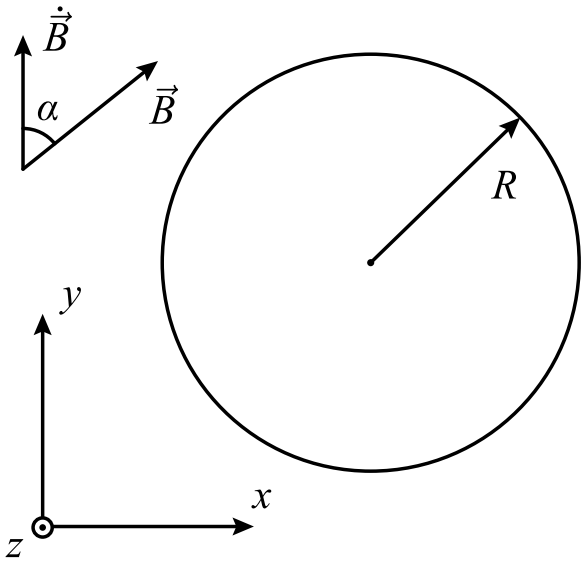


图 4：磁场中的球形壳

C.1	忽略自感，求出壳的感应磁矩 $\vec{\mu}$。将磁矩 $\vec{\mu}$ 的答案写成 xyz 分量形式（参见图 4）	1.0pt

C.2	求出球形壳上的力矩 $\vec{M}$. 将力矩 $\vec{M}$ 的答案写成 xyz 分量形式（参见图 4）.	0.3pt

地球磁场作用下的运动状态演化

现在我们研究火箭残骸（rocket stage）转动状态如何改变，这里假设火箭残骸轨道为圆形极地轨道，周期为 $T = 100\,min$(分钟)（参见图 5)。地磁场与火箭残骸的相互作用特征时间远大于暂态过程（transient process）持续时间。我们现在研究的是暂态过程完成之后，火箭残骸会发生什么。假设开始时火箭残骸绕垂直于轨道平面的转轴转动，角速度为 ω_2.

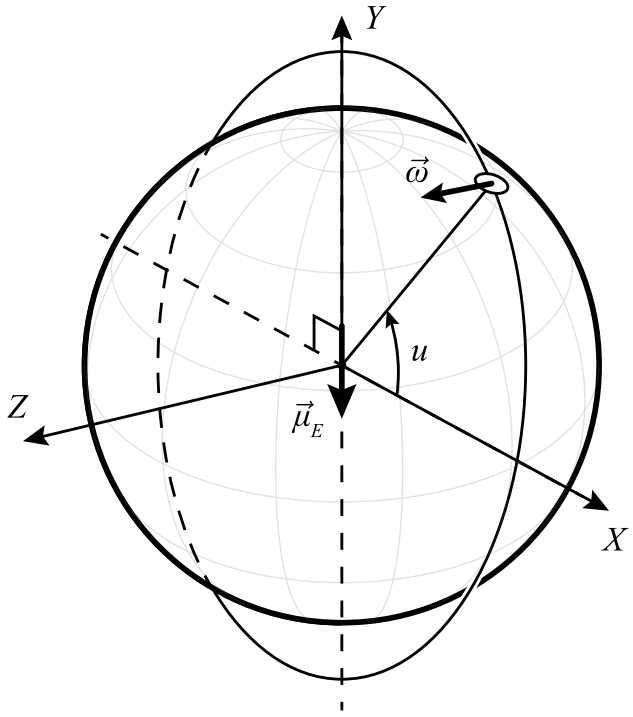


图 5：轨道

C.3 地球磁场 $\vec{B}_E$ 可以模型化为一个位于地球中心的点状磁偶极子产生的磁场。该磁偶极子磁矩 $\vec{\mu}_E$ 沿 Y 轴反方向。残骸轨道与赤道平面 XZ 交点处的地球磁场 B 的绝对值是 $B_0 = 20\,\mu T$（微特斯拉）。求出图 5 所示情况中，残骸轨道上角度为 u 处的磁场 $\vec{B}_E(u)$。轨道运动方向是角度 u 的正方向。将答案写成 $\vec{B}_E(u)$ 在 XYZ 方向上的投影分量形式。 0.4pt
注意：将 $\vec{B}_E(u)$ 的各分量表示成 $2u$ 的函数，而不是 u 的函数，可能会有助于后继的计算。
注意：磁偶极子在 $\vec{r}$ 处的磁场如下

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{\mu}}{r^3}\right). \tag{1}$$

“科博丁 42 号”的残骸大部分用木头制造，仅有低温燃料箱由导电材料制造。因此，我们将地磁场对残骸的作用等效于作用在一个壁厚 $D = 2\,\mathrm{mm}$、$R = 4\,\mathrm{m}$、电阻率 $\rho = 2.7 \cdot 10^{-8}\,\Omega\cdot\mathrm{m}$ 的球壳上。

C.4 当残骸绕垂直于轨道平面的转轴以角速度 ω 转动，且角速度方向沿 Z 轴方向时，求出作用在残骸上的力矩 $\vec{M}(u)$。将答案写成 M 在 XYZ 方向上的投影分量形式。 1.3pt

C.5 假设残骸的角速度在一个轨道周期内的变化量是可以忽略的小，求出角速度 $\omega(t)$ 作为时间函数的绝对值。 1.0pt

C.6 经过很长时间，进入稳态运行区时，求出轨道周期 T 与火箭残骸转动周期 T_s 的比值. 1.0pt

光子晶体 (20 分，5 小时)

简介

光子晶体是折射率具有周期性变化的材料，其周期性尺度与光波波长相当。在光子晶体的光谱中，存在一些抑制某些波长的光传播的狭窄带隙。这些特殊的光学特性，使光子晶体可应用于各种光学元件 (例如：光过滤器、反射器) 的开发上。

在这实验中，你将利用阳极氧化铝所制成的多孔膜 (包含着充满空气的孔) 来研究光子晶体的性质。通过电化学反应 (阳极化) 制成的阳极氧化铝多孔膜，其结构可看作一些垂直排列于薄膜上的不相交的圆柱通道 (图 1)。

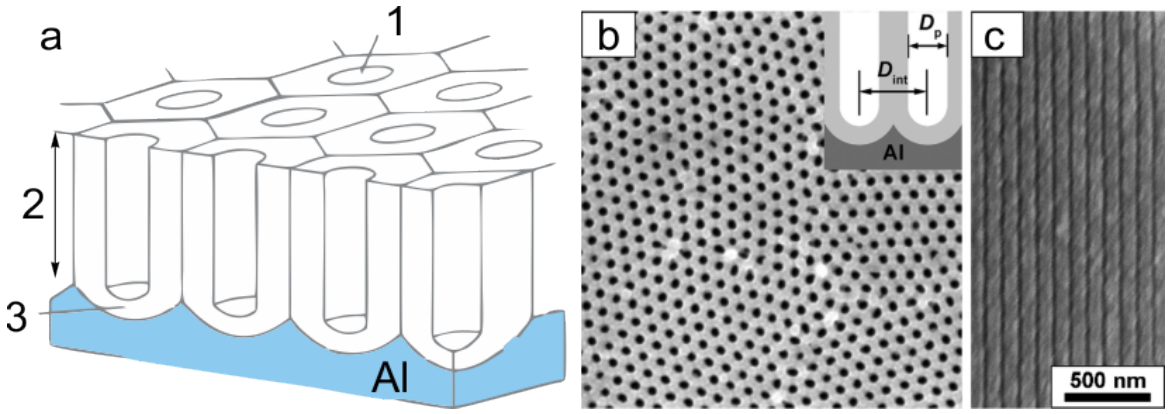


图 1：(a) 阳极氧化铝多孔膜结构图: (1) 孔、(2) 孔膜层、(3) 阻隔层。电子显微镜下的多孔膜图像：(b) 俯视图 (c) 横切面 [Electrochim. Acta, 2011, 56, 2378]。

孔的直径和孔间的距离，取决于电化学处理的情况，改变阳极化过程中的电压，可以得到在沿氧化铝膜表面法线方向上具有可变孔隙率的结构 (图 2)。

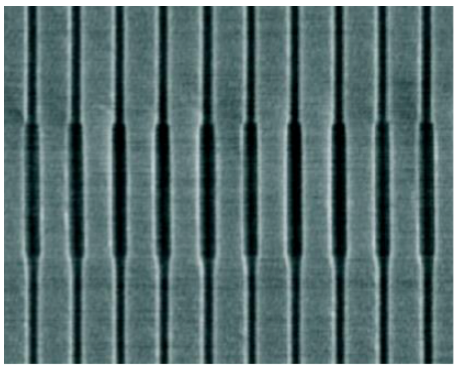


图 2：在电子显微镜下阳极氧化铝孔膜层的孔径变化的横切面图像 [Nat.Mater., 2006, 5, 741].

实验中各样品的孔径均小于 30 nm。具有这么小孔径的多孔介质在光学范围内的电磁波可视为连续介质，因此我们可利用等效（体积平均）折射率来表示它的折射率。

阳极氧化铝孔膜层的折射率变化是由孔隙率改变而造成的。注意实验样品的折射率只沿单一方向 (垂直于铝膜表面) 作周期性变化。因此实验样品是一维光子晶体。

应指出的是，均匀孔径的阳极氧化铝膜，在光学上是透明的，而光子晶体由于层结构造成光干涉，会呈现色彩。在这实验中，你将会研究三个利用阳极氧化铝 (AAO) 制成的不同结构的光子晶体样品。

仪器

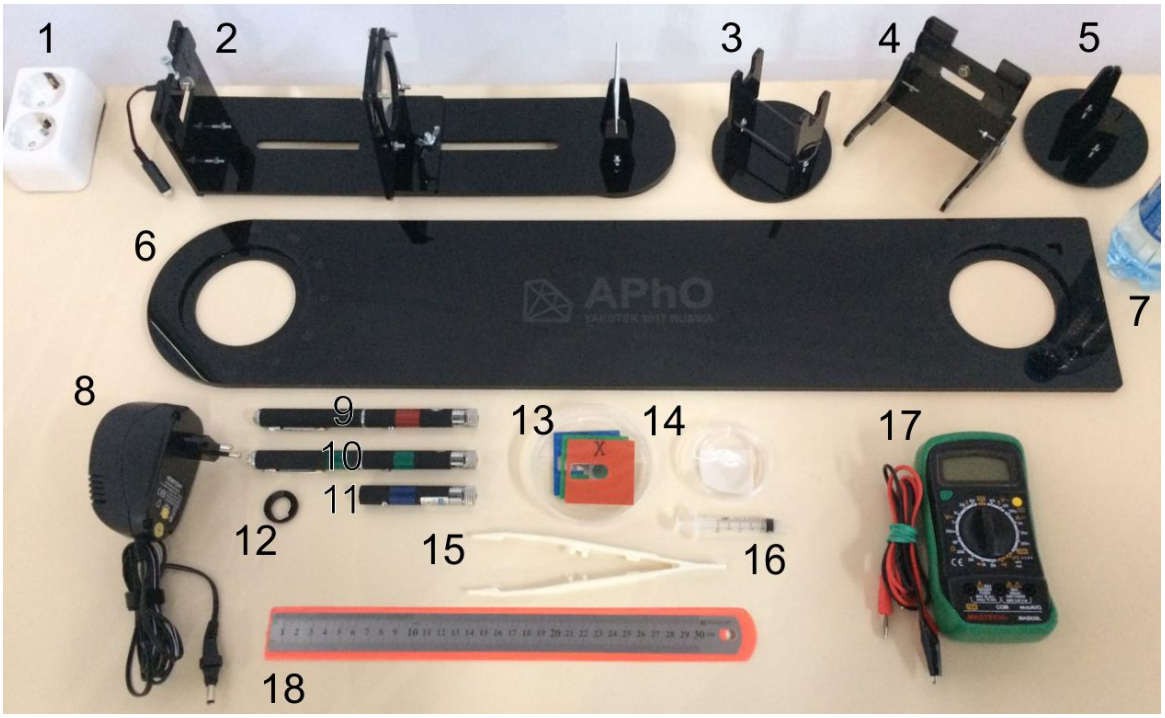


图 3. 桌上的仪器

1. 插座
2. 光谱测量用的衍射臂
3. 附有橡皮筋的激光支架
4. 光电二极管及其支架
5. 样品支架
6. 附有量角器的主光学平台
7. 瓶装水
8. 直流电源适配器 (提供电源)
9. 红色激光器 (波长为 659 nm)
10. 绿色激光器 (波长为 530 nm)
11. 蓝色激光器 (波长为 400 nm)

激光很危险！在任何情况下，切勿将激光射向眼睛或直视激光光束！亦须注意反射光束！

12. 激光开关按钮锁定圈
13. 培养皿中放置的 3 片光子晶体样品，分别装在不同颜色的框架中

光子晶体很易碎！切勿触摸晶体！只可手持框架！不会提供新样品！

14. 培养皿中放置的盖玻片

盖玻片很薄及易碎！使用镊子来夹起盖玻片！

15. 处理盖玻片的镊子
16. 针筒
17. 附有连接导线的万用电表
18. 直尺

衍射臂

在光谱测量中，需要用衍射臂 (图 4) 产生不同波长的光。衍射臂由白炽灯、在支架上的衍射光栅及在白炽灯与衍射光栅之间可移动的透镜组成。当安装衍射支架时，将圆盘放入主光学平台上左方的空洞里。

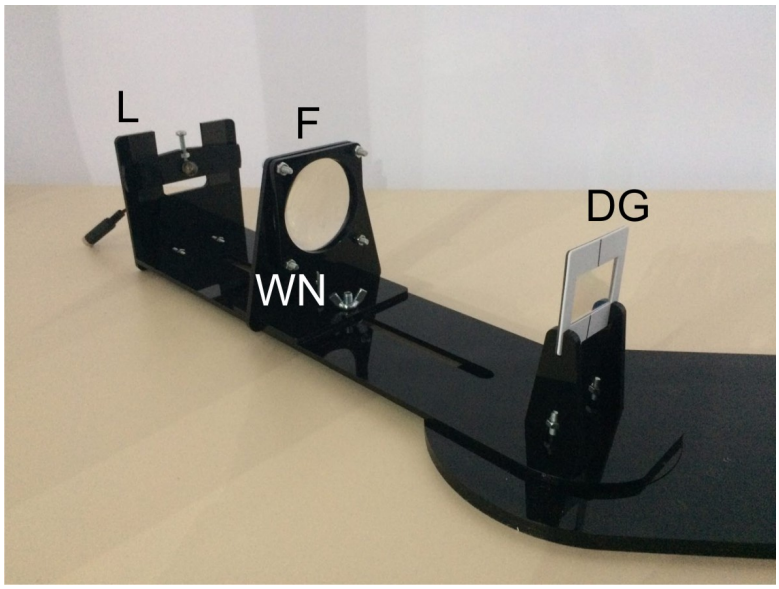


图 4. 衍射臂

白炽灯起初是置中的，然而如有需要你可自行调校。确保灯丝竖直以透过衍射光栅得到更精细的谱线。

移动透镜以调整光束会聚的角度。拧松翼形螺丝 (WN)，滑动透镜支架，然后再拧紧螺丝 (切勿过份拧紧，只需轻微扭矩以避免透镜不慎滑动)。

打开白炽灯及激光电源

将直流电源适配器插入桌上插座给光源供电。将直流电源适配器与白炽灯或激光器相连。要留意激光的正负极，起初的设定是正确的。

激光

提供了三款激光：红色激光 ($\lambda = 659$ nm)、绿色激光 ($\lambda = 530$ nm) 及蓝色激光 ($\lambda = 400$ nm)，用对应颜色的胶纸标示。

激光很危险！在任何情况下，切勿将激光射向眼睛或直视激光光束！亦须注意反射光束！

红色及蓝色激光共用同一供电装置。如需将红色转为蓝色激光，扭松红色激光的供电装置，然后把蓝色激光装上供电装置上 (图 5a)。

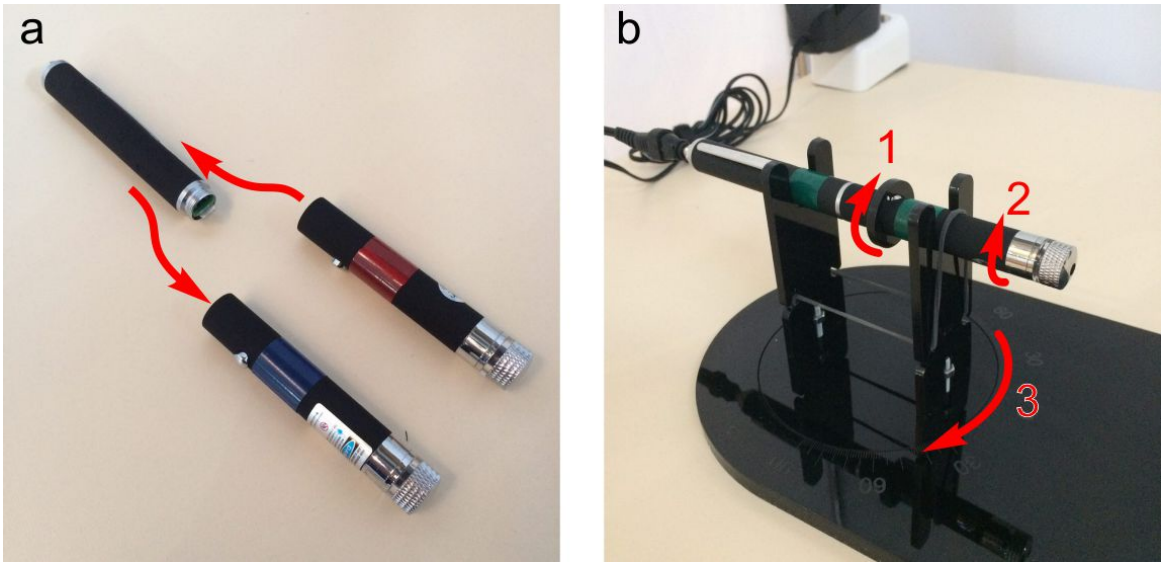


图 5：(a) 切换供电装置。(b) 调校激光光束。

在光度计的实验中，套上激光开关按钮锁定圈，并将激光器放在支架上，然后用橡皮筋缚紧 (图 5b)。将附有支架的圆盘放入主光学平台上左方的空洞里。转动激光开关按钮锁定圈以开启激光 (1)。

你可能需要校准激光光束。首先围绕激光器轴线转动激光器，直至光束在水平面内 (2)。然后，旋转附有支架的圆盘使得光束沿光学平台方向 (3).

光电二极管

在测量光强的实验中，你需要用到激光及光电二极管测量光的透射强度。光电二极管支架的支撑部份可插入主光学平台上的狭槽中 (图 6a)。

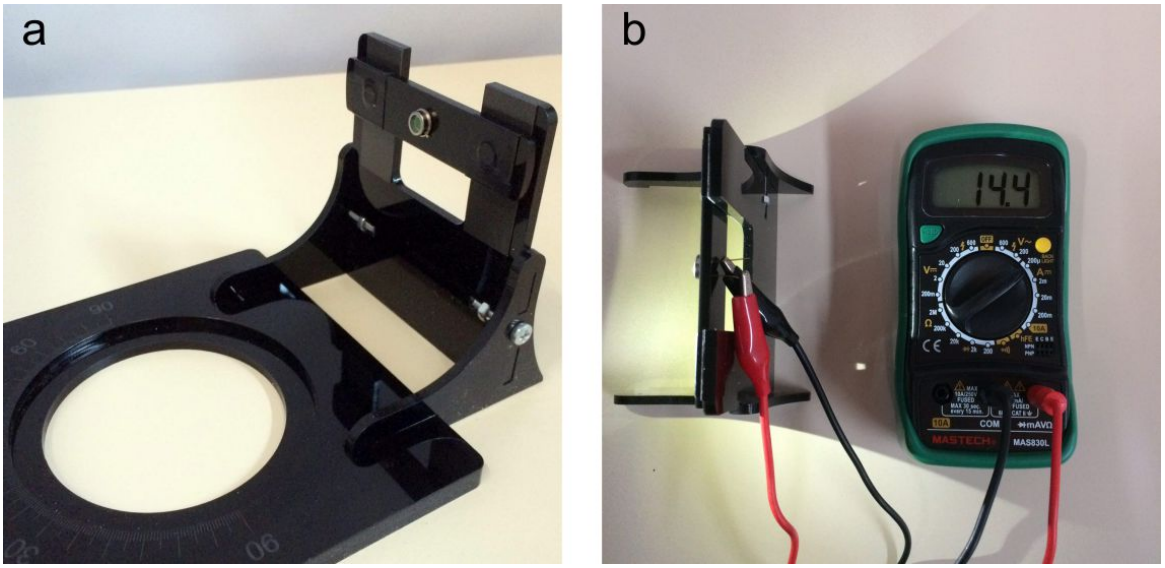


图 6：(a) 安装光电二极管。(b) 连接万用电表。

光电二极管安装在附有磁铁的支架上，可以根据光束方向作调校。

当使用光电二极管测量光强时，需将光电二极管连接万用电表，并使用直流电模式 (μA) (图 6b)。光电二极管的电流与所接收光的光度成正比，因此以 μA 为单位来测量光强。

光子晶体样品

光子晶体很易碎！切勿触摸晶体！只可手持框架！

在实验中，提供了三种不同的光子晶体样品。三种光子晶体样品的正面各有不同颜色，上面有样品名称 (X、Y 或 Z)。当将样品放上支架时，需确保光束射向样本的正面，而英文字母朝上。实验进行时，将附有样本的支架圆盘放入主光学平台上右方的孔洞里。注意需沿顺时针方向旋转附有样本的支架圆盘以避开反射光束。

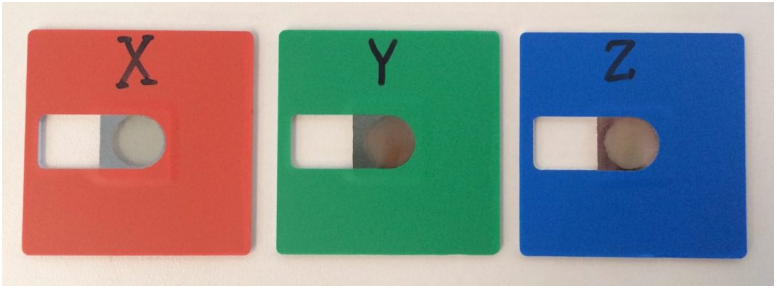


图 7：三种不同的光子晶体样品

布拉格-斯涅尔定律

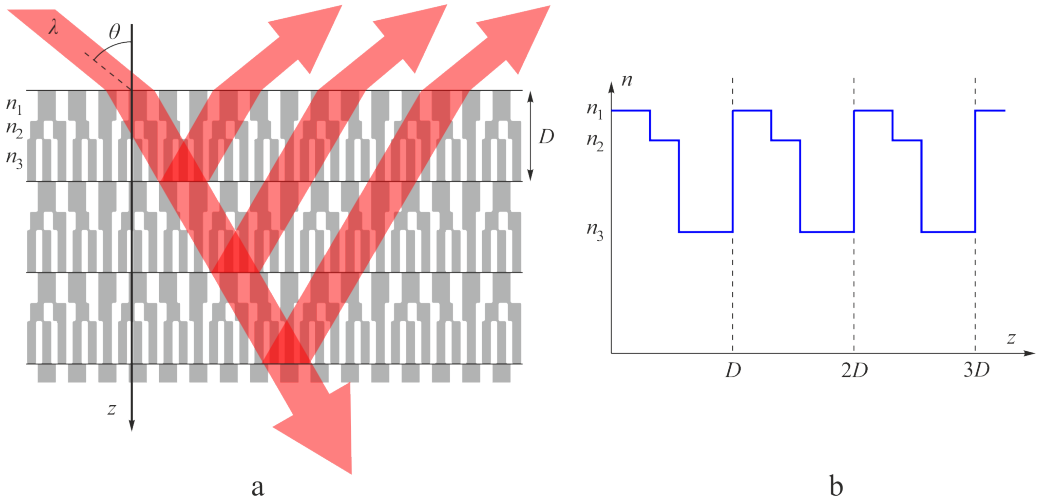


图 8. (a) 光子晶体结构. (b) 折射率 $n(z)$ 的变化。

本实验中的光子晶体样品拥有多层不同折射率 n_i 的层。折射率以周期 D 沿着 z 轴变化，但与波长 λ 无关。假设整体平均折射率记为 n，折射率差记为 $\Delta n = n_{\text{max}} - n_{\text{min}}$. 在 AAO 晶体中

$$\Delta n \ll n. \tag{1}$$

我们假设平行光束是单色波长 λ 及强度恒定地投射于光子晶体上。入射角是 θ . 从不同层反射的光束互相干涉。因此，在满足以下方程的入射角 θ，镜面反射有最大值：

$$2D\sqrt{n^2 - \sin^2\theta} = m\lambda, \tag{2}$$

其中 $m = 1, 2, \ldots$ 是表示干涉级次的整数。

公式 (2) 被称为**布拉格-斯涅尔定律 (Bragg-Snell law)**. 在反射率最大值的角度 θ_i 下，透射率则是最小值。若在角度 θ 不变的情况下改变波长 λ，布拉格-斯涅尔定律便可理解为一个随波长改变的方程式。图 9 显示了反射光谱和透射光谱。每个透射极小都对应着布拉格-斯涅尔定律中的某个整数 m .

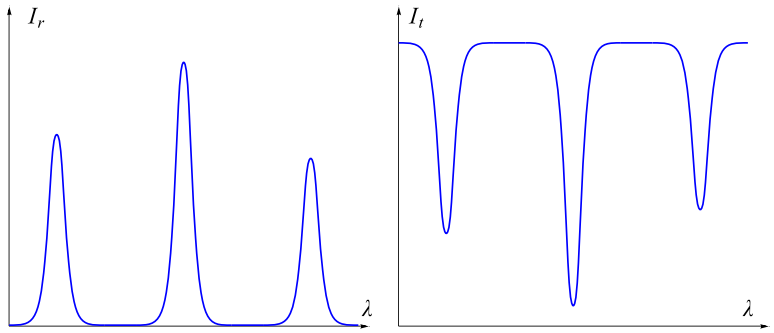


图 9：反射光谱和透射光谱。

满足布拉格-斯涅尔定律 (公式 2) 的波长被称为**透射极小波长 (wavelengths of transmittance minimums)**。极小透射波长取决于入射角 θ. 在 $\theta = 0$ 的情况下的极小透射波长则被称为该光子晶体的**透射极小的正入射波长 (normal wavelengths of transmittance minimums)**。

在本次实验考试中你不需要计算误差。

Part A . 样品 X. 光谱测量 (3.5 分)

样品 X 具有简单结构 (如图 10). 由折射率 (n_1 和 n_2) 相近而厚度同为 $D_X/2$ 的两层构成周期结构.

$$n_1 - n_2 = \Delta n \ll n_X = \frac{n_1 + n_2}{2} \tag{3}$$

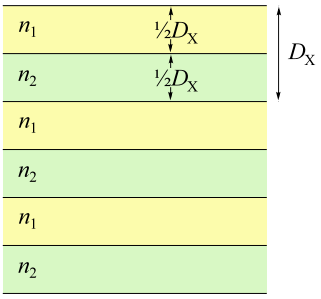


图 10 : 样品 X 的结构

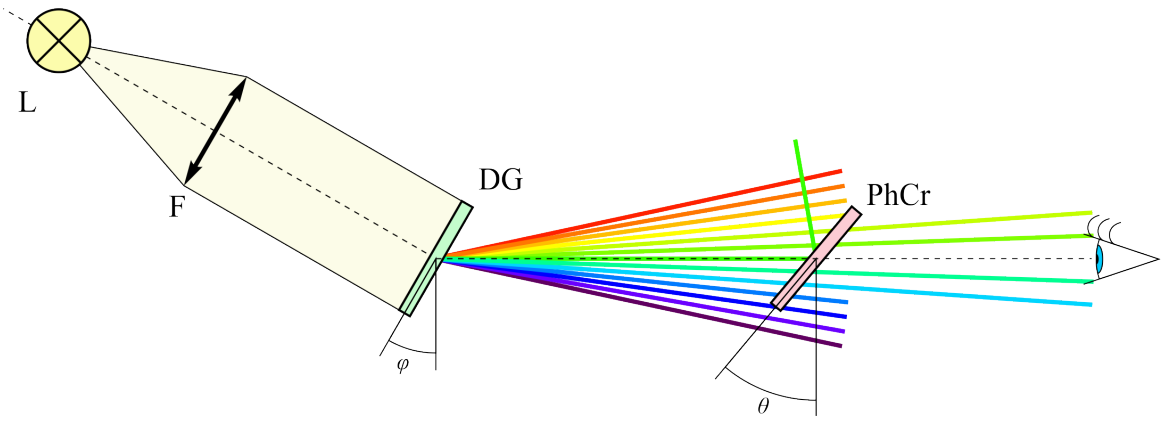


图 11 : 光谱测量的实验装置图：(F) 透镜, (L) 置于透镜焦点上的灯泡, (DG) 衍射光栅, (PhCr) 光子晶体.

1. 组装**用于光谱测量的实验装置**. 把衍射臂放入主光学平台的左孔洞内.

2. 打开灯. 从光学平台的另一末端观察衍射光栅. 转动衍射臂直到你看见第一阶衍射 (一组彩虹条纹).

3. 把样品支架放置在主光学平台的右孔洞内 (把三角标记设在 0 度). 把样品 X 放入支架内 (灯光应投射在红色一面上). 透过样品 X 去观察衍射光栅的彩虹条纹.

4. 如果顺时针转动样品到合适的角度, 你可看见透射光谱的黑暗区 (如图 12). 这个**透射极小 (transmittance minimum)** 对应于布拉格-斯涅耳 定律 (Bragg-Snell law) 中 $m = 1$. 当你转动样品时, 透射极小所对应的波长会改变.

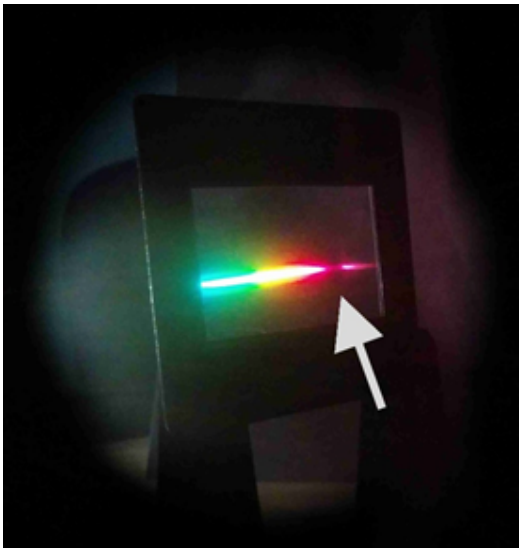


图.12. 样品 X 的透射光谱图. 白色箭头指出了透射极小 (彩虹条纹中的暗区).

5. 通过移动透镜，你可以缩放光谱以看到更宽或更窄的部分. 为了精确测量，请适当放置透镜，使得灯泡在焦点上.

A.1	光垂直于光栅表面入射. 写出联系衍射角 φ 和波长 λ 的方程. 光栅周期 $h = 1000$ nm（纳米）. 写出该答案时不需要推导过程.	0.1pt
A.2	测量对应透射极小值时的入射角 θ，测量时将其作为衍射角 φ 的函数. 注意读 θ 时, 要求透射极小区 (暗区) 位于光栅窗口的正中.	1.0pt
A.3	为了作出线性关系的图，请推导新的坐标关系以代替 θ 对 φ 的依赖关系. 用新坐标关系作图. (Find new coordinates instead of θ on φ to test that graph would be linear. Plot the graph in new coordinates.)	1.5pt
A.4	定出样品的周期 D_X 和平均折射率 n_X.	0.9pt

Part B. 样品 X. 激光测量 (5 分)

晶体的孔隙率 p, 是指晶体体积中被空气填充的部分占比率 (孔道). 晶体的等效折射率可以用空气折射率 $n_a = 1$ 和氧化铝折射率 n_{AAO} 表示为:

$$n_{dry} = \sqrt{pn_a^2 + (1-p)n_{AAO}^2}. \tag{4}$$

如果孔道中充满了水, 那么晶体的折射率会变为:

$$n_{wet} = \sqrt{pn_w^2 + (1-p)n_{AAO}^2}, \tag{5}$$

其中 $n_w = 1.33$ 是水的折射率.

在本部分题中，我们要对比样品干 (dry) 和湿 (wet) 两种情况下的实验数据，来定出光子晶体的孔隙率 p . 我们将使用激光测量的装置.

B.1	挑选激光器，使得对应激光波长 λ ，可以在某个样品转动角度 θ 时，能观察到透射极小. 写下你选出的激光的波长.	0.1pt

在 Part B 的所有任务中都使用已选出的激光.

组装**激光测量实验装置** (图 13).

不要直视激光束！注意跟踪反射光束, 不要射入眼睛. 即使短时的射入也可能对眼睛造成严重伤害. 做任何激光实验时都不要让头与光束持平.

激光实验中, 在记录数据之前, 要确保激光强度稳定为常数. 如果不稳, 等待 3-5 分钟.

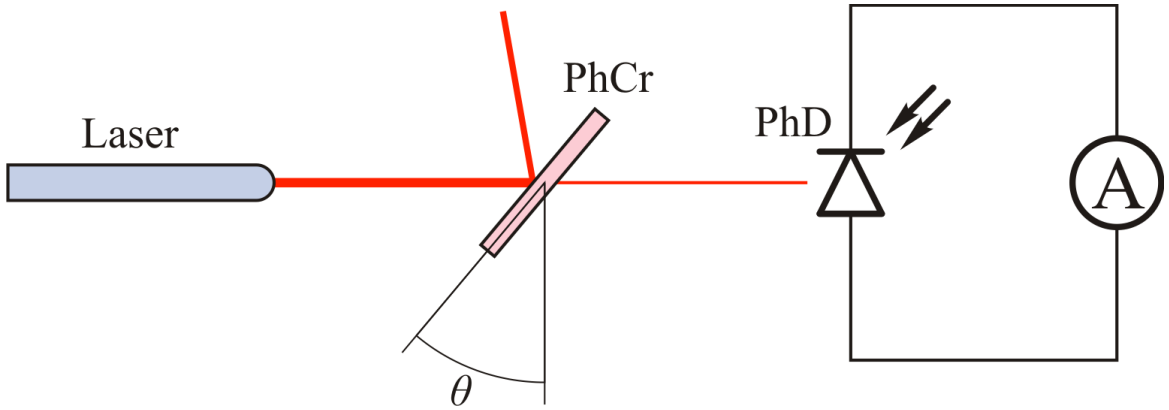


图 13. 激光测量装置图: (L) laser 激光, (PhCr) photonic crystal 光子晶体, (PhD) photodiode 光二极管, 连接到一个安培表. 安培表测量到的电流正比于落在光二极管上的光强.

B.2	测量透过光子晶体的激光光强 I_t (以 μA (微安) 为单位), 测量时将其作为 θ 的函数.	1.0pt

B.3	作图 $I_t(\theta)$.	1.0pt

B.4	定出此激光波长下，对应透射极小值的入射角 θ_1. 定出 $I_t(\theta)$ 图上波谷的半值宽度 $\Delta\theta_1$ (参见图 14a).	0.2pt

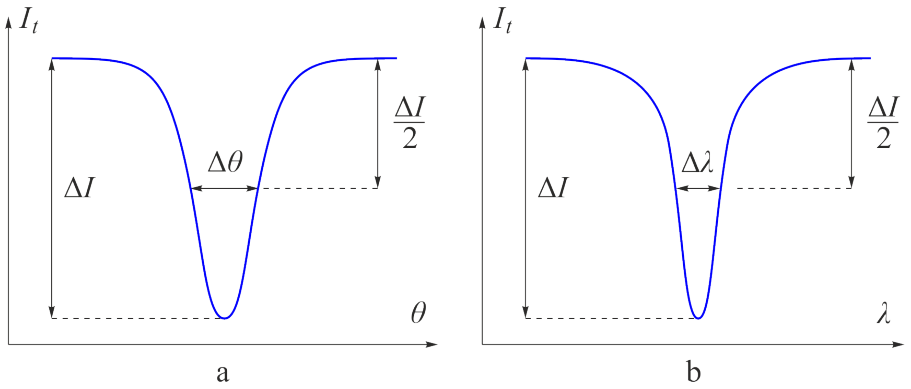


图 14. 透射光强波谷的半值宽度 (a) 关于入射角, (b) 关于波长.

B.5	用布拉格-斯涅尔定律 (Bragg-Snell law) (公式 2), 定出样品 X 的透射极小正入射波长 (transmittance minimum normal wavelength) λ_X . 使用 Part A 中得到的 n_X, B.4 中得到的 θ_1 , 以及激光波长 λ.	0.2pt

当光垂直于晶体入射时 ($\theta = 0$), 对应 $m = 1$ 的透射极小的光谱半值宽度 $\Delta\lambda$ (图 14b) , 可以用 n 和 Δn 表示为:

$$\frac{\Delta\lambda}{\lambda} = \frac{2}{\pi}\frac{\Delta n}{n}. \tag{6}$$

为了计算样品 X 的 Δn_X, 假设透射极小的相对光谱宽度 $\frac{\Delta\lambda}{\lambda}$ 不随转动而改变.

B.6	计算样品 X 的折射率差值 Δn_X .	0.6pt

从样品 X 朝上一面加入一些水. 为了防止实验过程中水的蒸发, 请非常小心的将一个盖玻片盖在样品上. 在样品的朝下一面，有一层 15 nm(纳米) 厚的固体氧化物阻止水的流出.

请当心！样品非常脆弱。样品的宽度只有一根头发直径的四分之一。操控样品时只能接触边框。盖玻片同样也非常脆弱！只能使用夹子来夹持盖玻片。

B.7	根据激光波长下湿样品 (the wet sample) 的透射极小值定出入射角 θ_2	0.3pt

B.8	定出样品 X 的平均孔隙率 p_X 和氧化铝的折射率 n_{AAO}。	1.0pt

B.9	定出样品 X 各层的平均孔隙率 p_1 和 p_2。	0.6pt

Q1-13

Part C. 样品 Y。多个透射极小（4.5 分）

样品 Y 的结构比样品 X 的结构要更复杂。样品 Y 在可见光波段 (400-800 nm) 有 4 个透射极小值，分别对应着布拉格-斯涅耳定律 (公式 2) 中 4 个连续的整数 m. 样品 Y 和样品 X 的折射率相同 $n_Y = n_X$.

C.1	虽然样品 Y 有 4 个透射极小值，但在光谱测量中你只能观察到其中 3 个。采用光谱测量 (见 Part A)，定出这 3 个透射极小值所对应的正入射波长 $\lambda_1^{sp}, \lambda_2^{sp}, \lambda_3^{sp}$.	0.6pt

对样品 Y 进行激光测量 (见 Part B)。激光测量能让你以更高的精度定出透射极小值所对应的正入射波长。此外，你还能定出透射光强 (在 part E 中将会用到)。

C.2	对红色激光，测量透射光强 I_{red} 随样品旋转角度 θ 的变化曲线。	0.5pt
C.3	对绿色激光，测量透射光强 I_{green} 随样品旋转角度 θ 的变化曲线。	0.5pt
C.4	对蓝色激光，测量透射光强 I_{blue} 随样品旋转角度 θ 的变化曲线。	0.5pt
C.5	利用任务 C.2-C.4 中获得的数据，定出 4 个透射极小值所对应的正入射波长值。	0.6pt
C.6	定出 C.5 任务中 4 个正入射波长所对应的整数 m. 如果需要的话你可以画一幅图。	1.0pt
C.7	定出样品 Y 的周期 D_Y (以 nm 为单位)。	0.2pt

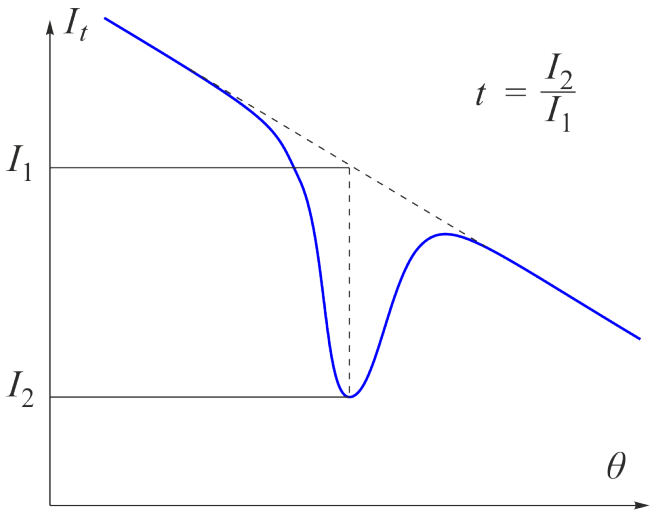


图 15. 利用 $I(\theta)$ 曲线定出 t

C.8	定出样品 Y 的 4 个透射极小值所对应的这些 t (图 15)。 这些值将会在 part E 中被用来确定样品 Y 的结构。如果需要的话你可以画图，尽管题目没有要求这么做。	0.6pt

Part D. 样品 Z. 缺失的透射极小 (4.5 分)

布拉格-斯涅耳定律 (公式 2) 决定了在哪些入射角和波长下透射极小能被观察到。t 取决于样品周期的内在结构。如果样品具有特定的结构，布拉格-斯涅耳定律所预言的某些透射极小值具有 $t = 1$，因而不能被观察到。我们称这些透射极小为**缺失的透射极小 (missed transmittance minimums)**。对那些没有缺失的极小，我们称之为**可见的 (visible)**。

在光谱实验或激光实验中，如果在布拉格-斯涅耳定律所预言的位置没有看见透射极小，我们就认为这些极小是缺失的。样品 Y 没有缺失的透射最小。相反，样品 Z 在可见光波段有 2 个缺失的极小值。这意味着所有可见的透射极小值并不是对应着连续的整数 m.

样品 X 和样品 Z 的平均折射率相同: $n_Z = n_X$.

D.1	定出样品 Z 的可见的透射极小值所对应的正入射波长 λ_Z. 进行光谱实验和激光实验。利用简图和公式来描述你的实验。	1.2pt
D.2	定出对应可见透射极小值的整数 m. 如果需要的话你可以画一幅图。	2.0pt
D.3	定出样品 Z 的周期 D_Z (以 nm 为单位)。	0.3pt
D.4	定出哪几个透射极小值是缺失的。推导它们在布拉格-斯涅耳定律中的 m' 和正入射波长 λ'_Z.	1.0pt

Part E. 样品 Y 和 Z. 周期的内部结构. (2.5 分)

布拉格-斯涅耳定律（公式 2）描述了实验中观察到反射极大和透射极小时所对应的角度值和波长值（图 9）。透射值依赖周期的内部结构。

这部分将讨论反射值的简单理论。这里将考虑光的正入射 ($\theta = 0$)，由此情形可将布拉格-斯涅耳定律简化为：

$$2Dn = m\lambda \tag{7}$$

这里 D 为晶体的周期.

在这里的模型中，我们假设光只有从高折射率介质层向低折射率介质层传播时才在界面上反射。模型中的反射界面在图 16 中用箭头标出。

样品 X、Y、Z 都由相同厚度、两种不同的折射率的材料组成，因此，我们将只限于这种特殊情况。

令 d_l 为双介质层的厚度，D 为晶体的周期, δ_j 为周期 (period) 开始位置到反射界面 j 之间的距离.

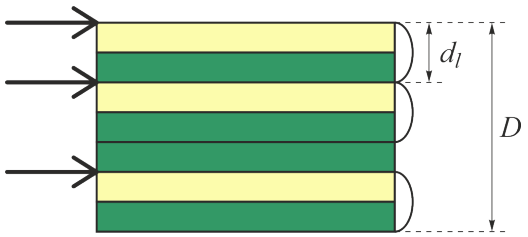


图 16. 现以周期为 D 的样品的结构为例. 深绿色层的折射率大，黄色层的折射率小，每层的厚度均为 $d_l/2$. 这里模型所考虑的反射界面均用箭头标出.

例如，图 16 的晶体结构的周期为 $D = 3.5d_l$, 到反射界面的距离分别为 $\delta_1 = 0, \delta_2 = d_l, \delta_3 = 2.5d_l$.

对于从第 j 层反射的光束，其相位增加值 (phase advance) 为

$$\varphi_{j_m} = 2\pi\frac{2\delta_j n}{\lambda} = 2\pi m\frac{\delta_j}{D} \tag{8}$$

考虑从同一个周期反射的光束的干涉，我们导出对应于布拉格-斯涅尔定律中 m 级反射极大的反射强度：

$$I_{refl,m} \sim \left|\sum_j \exp(i\varphi_{j_m})\right|^2 \tag{9}$$

这里 i 为虚数单位.

上述模型不是定量正确的。但它反映了多个反射极大的顺序：你可以比较这些极大值，并确定哪个强度更大。如果反射极大较强，相应的透射极小的 t 就较小。如果反射极大足够小，我们就认为这是缺失的透射极小。

在附录中，根据公式 (9) 计算了从 1 到 20 的整数 m 对应的 I_m/I_0 值，并列于表中。I_0 是根据 (9) 式计算的 $m = 0$ 时的强度，此时，所有反射光束在没有相移的情况下发生干涉，使最大强度成为可能。这里的 I_m/I_0 表示成百分数（%）.

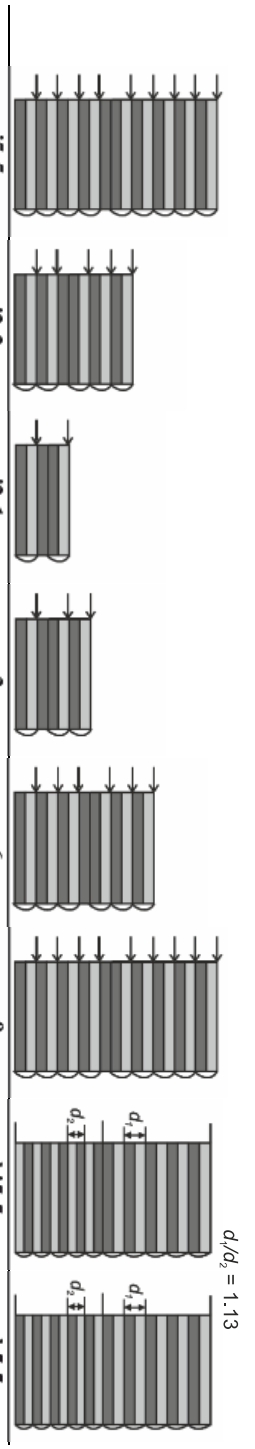
样品 Y 和 Z 的结构也列于表中。

E.1	将从 C.8 中获得的样品 Y 的透射值与附录表中的结果进行比较。确定样品 Y 的结构。把其结构的名称写在答题卡里。	1.2pt

E.2	确定样品 Z 的结构。把其结构的名称写在答题卡里。	1.3pt

附录

样品 Y 和 Z 的各种可能的结构. 图下方给出了各种结构的名称. 表中的数字是 I_m/I_0.



m	i5-5	i3-3	i2-1	n-3	n-6	n-9	hi5-5	h5-5
1	1	3	11	3	1	0	1	0
2	0	0	33	7	1	0	0	0
3	1	6	11	56	1	0	1	0
4	0	0	33	56	2	0	0	0
5	2	41	11	7	6	1	2	0
6	0	0	100	3	48	1	1	0
7	5	41	11	100	48	2	3	0
8	0	0	33	3	6	5	6	0
9	41	6	11	7	2	45	36	11
10	0	0	33	56	1	45	1	72
11	41	3	11	56	1	5	32	15
12	0	100	100	7	1	2	10	2
13	5	3	11	3	100	1	1	0
14	0	0	33	100	1	1	2	0
15	2	6	11	3	1	0	0	0
16	0	0	33	7	1	0	1	0
17	1	41	11	56	2	0	0	0
18	0	0	100	56	6	0	6	7
19	1	41	11	7	48	100	28	27
20	100	0	33	3	48	0	25	23

Vortices in Superfluid
MODB

2017-05-03

Introduction

Superfluidity is a property of flowing without friction. Everyday experience tells us that motion of an ordinary fluid (say, water at room temperature) is always accompanied by viscous dissipation of energy, so that the flow gradually becomes slower unless it is maintained by external forces. In contrast, superfluid exhibits no loss of kinetic energy: once excited the motion of superfluid can continue indefinitely. Superfluidity was originally discovered experimentally in liquid helium.

We study properties of superfluid helium at zero temperature. It will be treated as an incompressible fluid with density ρ. Flow continuity (the fact that the mass flowing into and the mass flowing out of a given infinitesimal volume are equal) implies that the flux of helium velocity $\vec{v}$ through a closed surface is always zero. Superfluid velocity in this aspect is similar to the magnetic field intensity. Similarly to the magnetic field lines, "streamlines" are tangential to the fluid velocity at each point and their density is proportional to its magnitude.

True superflow has an important property of being irrotational: circulation of superfluid velocity $\vec{v}$ along any closed path within helium is zero

$$\int_L \vec{v} \cdot d\vec{l} = 0.$$

This statement must be amended if superfluidity is absent along a thin "vortex filament". The thickness of the filament itself is of approximately atomic dimensions a, but the vortex induces long range velocity field in surrounding superfluid: velocity circulation around such filament is the circulation quantum[1]

$$\left| \int_L \vec{v} \cdot d\vec{l} \right| = 2\pi\kappa, \tag{1}$$

[1] Circulation quantization is a macroscopic quantum effect and corresponds to the angular momentum quantization in Bohr model. The circulation quantum can be expressed as $\kappa = \hbar/m_{\mathrm{He}}$, where m_{He} is the mass of helium atom.

and zero if the path can be contracted to a single point without crossing the vortex, see Fig. 1. This supports the analogy between superflow and magnetic field: superposition of two valid velocity distributions is a valid velocity distribution and the velocity at any point is equal (up to a dimensional factor) to the magnetic field produced by the unit currents running through a system of wires representing vortex filaments.

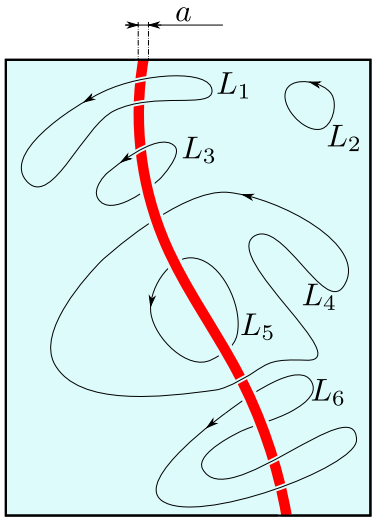


Figure 1: Vortex filament (red) in superfluid (light blue). Velocity circulations along paths L_1, L_2, L_5, and L_6 are all zero, but those for L_3 and L_4 are equal to $\pm 2\pi\kappa$. Note that circulations along L_3 and L_4 have opposite signs.

A. Steady filament (0.75)

Consider a cylindrical beaker (radius $R_0 \gg a$) of superfluid helium and a straight vertical vortex filament in its center Fig. 2.

A1 (0.25)

Plot the streamlines. Find out the velocity v at a point $\vec{r}$.

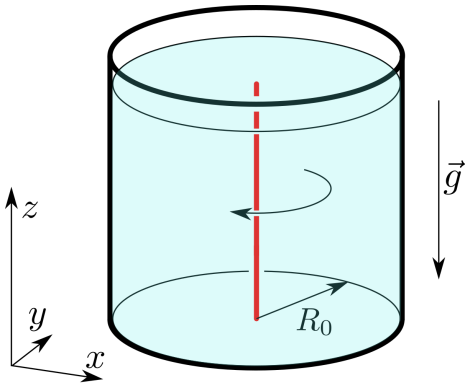
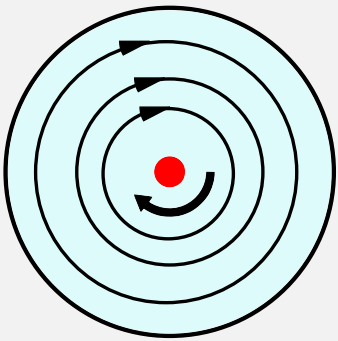


Figure 2: Straight vortex along the axis of a beaker.



The streamlines are circular. From the circulation identity (1) it is obvious that $v = \kappa/r$.

- Streamlines are plotted correctly 0.1
- $v = \frac{k}{r}$ 0.15

A2 (0.5)

Work out the free surface shape (height as a function of coordinate $z(\vec{r})$) around the vortex. Free fall acceleration is g, surface tension can be neglected.

Consider a thin circular layer of the radius r. Equilibrium condition for its surface is given by the requirement

$$g\frac{dz}{dr} = \frac{v^2}{r} = \frac{\kappa^2}{r^3}. \tag{2}$$

This equation is satisfied by the surface profile

$$z(r) = -\frac{\kappa^2}{2gr^2}. \tag{3}$$

- $\tan\alpha = \frac{k^2}{gr^3}$ or equivalent 0.25
- $z = -\frac{k^2}{2gr^2}$ 0.25

B. Vortex motion (1.25)

Free vortices move about in space with the flow[2]. In other words each element of the filament moves with the velocity $\vec{v}$ of the fluid at the position of that element.

As an example, consider a pair of counter-rotating straight vortices placed initially at distance r_0 from each other, see Fig. 3. Each vortex produces velocity $v_0 = \kappa/r_0$ at the axis of another. As a result, the vortex pair moves rectilinearly with constant speed $v_0 = \kappa/r_0$ so that the distance between them remains unchanged.

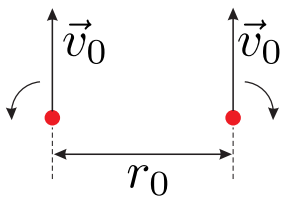


Figure 3: Parallel vortex filaments with opposite circulations

B1 (0.25)

Consider two identical straight vortices initially placed at distance r_0 from each other as shown in Fig. 4. Find initial velocities of the vortices and draw their trajectories.

[2]This is a consequence of momentum conservation, see next section.

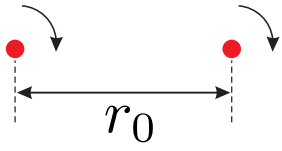
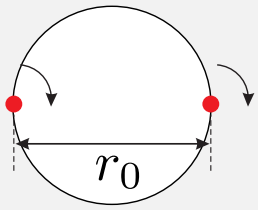


Figure 4: Parallel vortex filaments with equal circulations



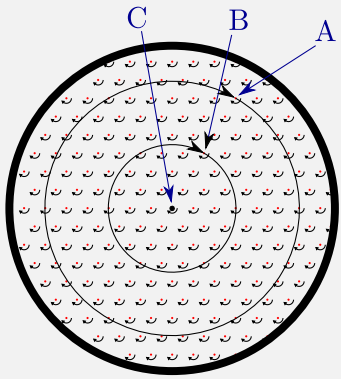
Being advected by each other's flow field, filaments will rotate around a point halfway between them. The velocity is given by $v_0 = \kappa/r_0$.

- Trajectories are plotted correctly 0.15
- Correct expression for velocity 0.1

A beaker of helium is filled with triangular lattice ($u \ll R_0$) of identical vertical vortices, see Fig. 5.

B2 (0.15)

Draw the trajectories of vortices A, B, and C (located in the center).



- Trajectories are plotted correctly (0.05 for each) 0.15

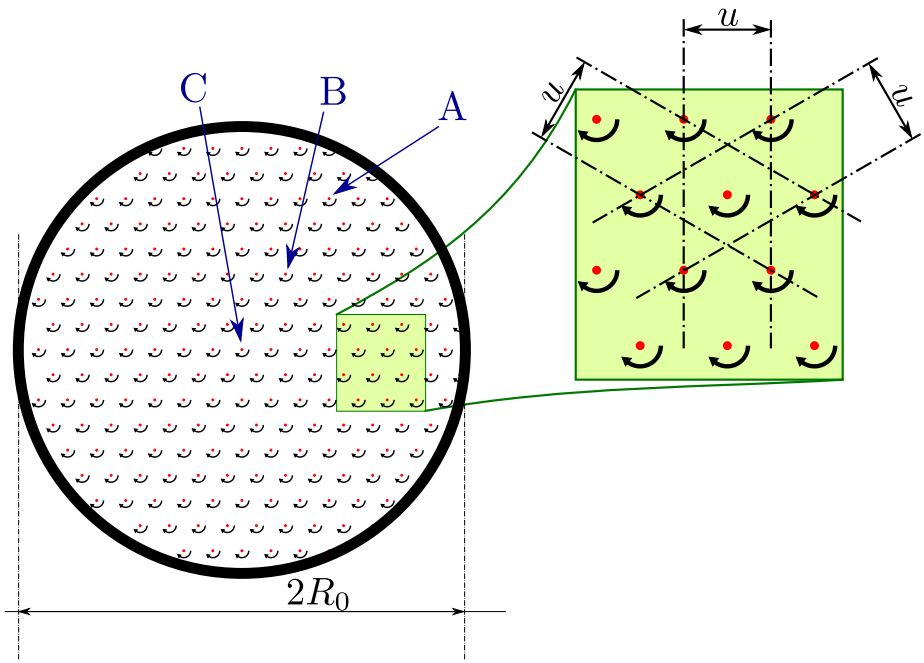


Figure 5: Triangular lattice of vortices in a beaker. The view from above.

B3 (0.4)

Find velocity $v(\vec{r})$ of a vortex positioned at $\vec{r}$.

Consider a circular path of radius $r \gg u$ around the beaker center. The circulation along this path is given by the number of vortices within it (vortex density per unit area is $(u^2\sqrt{3}/2)^{-1}$):

$$2\pi r v = 2\pi\kappa \frac{\pi r^2}{u^2\sqrt{3}/2}. \tag{4}$$

The velocity field

$$v = \frac{2\pi\kappa r}{u^2\sqrt{3}}. \tag{5}$$

- Expression for vortex density 0.2
- Correct expression for $v(r)$ 0.2

B4 (0.2)

Find the distance AB(t) between the vortices A and B at time t. Treat AB(0) as given.

This velocity pattern corresponds to the rotation of the lattice as a whole around the beaker center with angular velocity

$$\omega = \frac{2\pi\kappa}{u^2\sqrt{3}}. \tag{6}$$

AB(t) = AB(0)

- Correct answer 0.2

B5 (0.25)

Work out the "smoothed out" (omitting the lattice structure) free helium surface shape $z(\vec{r})$.

The surface shape is

$$z(r) = \frac{\omega^2 r^2}{2g} = \frac{4\pi^2\kappa^2 r^2}{6gu^4}. \tag{7}$$

- Correct answer 0.25

C. Momentum and Energy (1.75)

The long range velocity field is the major contribution to the energy of a system of vortices, it is insensitive to exact structure of the filament. The filament itself can not be properly described by the macroscopic theory and apparent singularities (infinities) are insignificant. Real physical quantities, such as the energy, of the region inside a thin tube of radius a around the filament should be neglected. Outside of this tube the density of superflow kinetic energy $\rho v^2/2$ (where $\rho = \text{const}$) is similar to the energy density of the magnetic field $B^2/(2\mu_0)$ — they are both quadratic in respective variables. This similarity facilitates calculation of the flow energy for a given system. For instance, given the inductance of a circular wire loop $L \approx \mu_0 R \log(R/a)$, where R is the loop radius and a is wire radius, we get the superfluid vortex loop energy[3]

$$U \approx 2R\rho\pi^2\kappa^2 \log(R/a). \tag{8}$$

Total fluid momentum is also determined by the long range velocity distribution. It is obtained by integration of the momentum density $\rho\vec{v}$. Again, consider

[3]This expression is also valid only if $\log R/a \gg 1$.

a flow generated by a circular vortex loop placed in xy plane. It is obvious from the symmetry considerations, that total momentum has only z component:

$$P = \int \rho v_z dV = \rho \int \int \underbrace{\left(\int v_z dz \right)}_{q(x,y)} dxdy \tag{9}$$

The innermost integration is in fact an integration along appropriate paths parallel to z-axis, see Fig. 6. From the circulation identity (1) it follows that

$$q(x,y) = \int\limits_{L(x,y)} \vec{v} \cdot d\vec{l}$$

is piecewise constant. Particularly, it is zero for paths passing outside the ring and $2\pi\kappa$ for paths inside it. Total momentum is therefore

$$P = \rho \cdot \pi R^2 \cdot 2\pi\kappa = 2\pi^2 \rho R^2 \kappa. \tag{10}$$

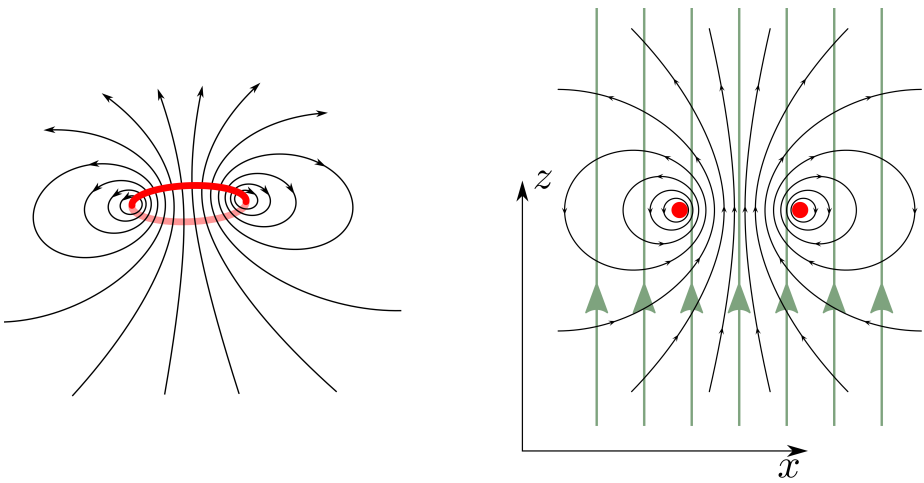


Figure 6: Velocity field of a circular vortex loop and integration paths (green) for $q(x,y)$ calculation.

C1 (0.3)

Consider a nearly rectangular vortex loop $b \times d$, $b \ll d$, Fig. 7.

Indicate the direction of its momentum $\vec{P}$. Find out the momentum magnitude.

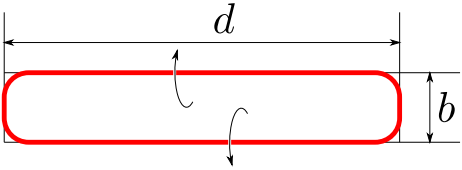
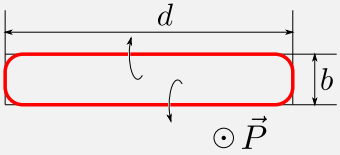


Figure 7: A nearly rectangular vortex loop, $b \ll d$



Momentum of a flat loop (see Introduction) is perpendicular to its plane and proportional to its area. For a rectangular loop the magnitude is $P = 2\pi\kappa\rho bd$.

- Correct direction of momentum 0.15
- Correct expression for momentum magnitude 0.15

C2 (0.7)

Calculate its energy U.

To produce equal magnetic and kinetic energy densities $B^2/(2\mu_0) = \rho v^2/2$, the magnetic field has to be $B = v\sqrt{\mu_0\rho} = \kappa\sqrt{\mu_0\rho}/r$. This field is generated by a current $I = 2\pi\kappa\sqrt{\rho/\mu_0}$. Energy of the wire loop can be found from the inductance $U = LI^2/2$. Inductance of a nearly rectangular wire loop:

$$L = \frac{\Phi}{I} = 2dI^{-1}\int_a^b \frac{\mu_0 I}{2\pi r}dr = \frac{\mu_0 d}{\pi}\log\frac{b}{a}. \tag{11}$$

This gives for the energy

$$U = 2\pi\kappa^2\rho d\log\frac{b}{a} \tag{12}$$

- Integration limits are a and b 0.2
- Analogy with a magnitude field is used ($U = \frac{LI^2}{2}$, $L = \frac{\Phi}{I}$) or energy is calculated as $W = \int F\,dr$, where $F = \frac{dP}{dt}$ 0.2
- Correct expression for energy 0.3

C3 (0.75)

Suppose we shift a long straight vortex filament by a distance b in x direction, see Fig. 8. How much does the fluid momentum change? Indicate the momentum change direction. The filament length (constrained by the vessel walls) is d.

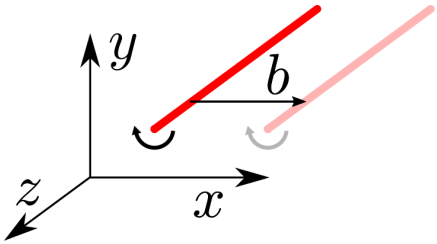
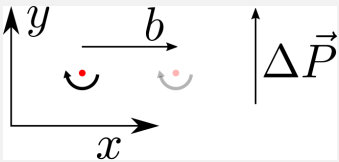


Figure 8: Momentum changes whenever the vortex shifts with respect to the fluid.



The momentum change is equal to the momentum of a long rectangular loop $P = 2\pi\kappa\rho bd$.

- The result of C1 used 0.3
- Momentum change is parallel to Y axis 0.1
- Correct direction of momentum change 0.15
- Correct expression for momentum change magnitude 0.2

Interestingly, this provides an alternative approach to find the energy of such a loop. Namely, if we slowly move one straight vortex in the velocity field of another, then we apply a force

$$F = 2\pi\kappa\rho dv = 2\pi\kappa\rho d\frac{\kappa}{r} = \frac{2\pi\kappa^2\rho d}{r}. \tag{13}$$

The work

$$W = \int_a^b \frac{2\pi\kappa^2\rho d}{r} dr = 2\pi\kappa^2\rho d \log\frac{b}{a} \tag{14}$$

has to be performed to move it from distance a to b.

D. Trapped charges (2.5)

Electrons, if injected in helium, get trapped by the vortex filaments. Here and below polarizability of helium can be neglected ($\epsilon = 1$).

D1 (0.5)

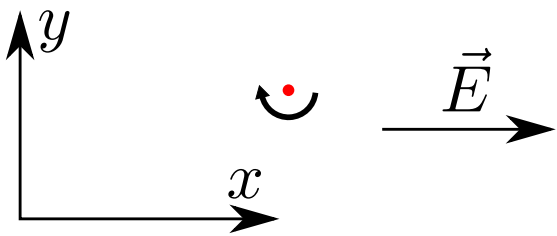
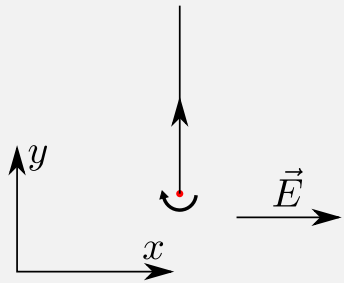


Figure 9: Straight vortex in a uniform electric field.

Consider a straight vortex charged with uniform linear density $\lambda < 0$ in a uniform electric field $\vec{E}$. Draw the vortex trajectory. Find its velocity as a function of time.



Electric force $F = E\lambda d$ moves the vortex with velocity

$$v = \frac{F}{2\pi\kappa\rho d} = \frac{E\lambda}{2\pi\kappa\rho} \tag{15}$$

perpendicular to $\vec{E}$.

- Trajectory is straight line parallel to Y axis 0.1
- Correct direction of velocity 0.2
- Correct expression for velocity magnitude 0.2

A circular vortex loop of radius R_0 initially charged with uniform linear density $\lambda < 0$ is placed in a uniform electric field $\vec{E}$ perpendicular to its plane, opposite to its momentum $\vec{P}_0$.

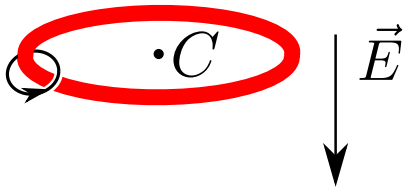
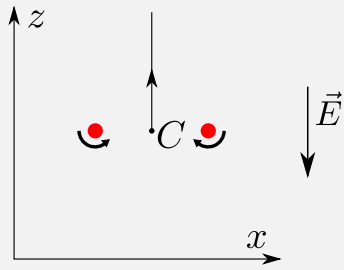


Figure 10: Vortex ring in a uniform electric field.

D2 (0.25)

Draw the trajectory of the loop center C.



- Trajectory plotted correctly 0.25

D3 (1.5)

Find its velocity $v(t)$ as a function of time.

Electric force upon the loop $F = 2\pi E R_0 \lambda$ is constant and fluid momentum linearly depends on time

$$P = P_0 + 2\pi E R_0 \lambda t = 2\pi^2 \rho R^2 \kappa. \tag{16}$$

The loop is growing and its radius is increasing with time t

$$R = \sqrt{R_0^2 + \frac{E R_0 \lambda t}{\pi \rho \kappa}}. \tag{17}$$

The loop velocity v can be easily found from a relationship between the energy change rate and the momentum change rate

$$\frac{dU}{dt} = Fv = \frac{dP}{dt} v. \tag{18}$$

This gives for the velocity

$$v = \frac{dU}{dP} \approx \frac{\kappa}{2R} \log \frac{R}{a} = \frac{\kappa \log \frac{\sqrt{R_0^2 + E R_0 \lambda t/(\pi\rho\kappa)}}{a}}{2\sqrt{R_0^2 + E R_0 \lambda t/(\pi\rho\kappa)}} \approx \frac{\kappa \log \frac{R_0}{a}}{2\sqrt{R_0^2 + E R_0 \lambda t/(\pi\rho\kappa)}}. \tag{19}$$

This means that the vortex is moving in the direction of the force but its velocity is decreasing.

- Correct expression for $R(t)$ 0.5
- Expression for $v \propto \frac{1}{R} \log(R)$ 0.5
- Correct expression for $v(t)$ 0.5

D4 (0.25)

The field is switched off at a time t^* when the velocity reaches the value $v^* = v(t^*)$. Find the loop velocity $v(t)$ at a later time $t > t^*$.

When $E = 0 \Longrightarrow P = const \Longrightarrow R = const \Longrightarrow v = const \Longrightarrow v(t) = v^*$.

- Correct expression for $v(t)$ 0.25

E. Influence of the boundaries (2.75)

Solid walls alter the velocity field created by a vortex filament, because the fluid cannot flow through them. Mathematically this means that the wall-normal

velocity component vanishes at the wall surface.

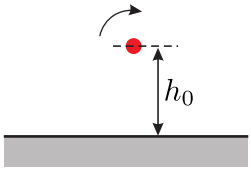


Figure 11: Straight vortex filament near a flat wall

E1 (0.5)

Draw the trajectory of a straight vortex, initially placed at a distance h_0 from a flat wall. Find its velocity as a function of time.

Well known technique of image charges (currents) in electrostatics (magnetostatics) can be directly used to solve this problem. Namely, the wall can be "substituted" with a reflected fictitious vortex on the other side of the wall. The velocity distribution of two vortices together in the upper semi-space is identical to the one produce by a single vortex above the wall. Indeed, the symmetry of the problem ensures that there is no flow through the plane of symmetry. Thus, a straight vortex line situated a distance h_0 above a flat wall with its image behave as a pair of vortices of opposite circulation a distance $2h_0$ apart. This means that the vortex moves along the wall with velocity

$$v = \frac{\kappa}{2h_0}. \tag{20}$$

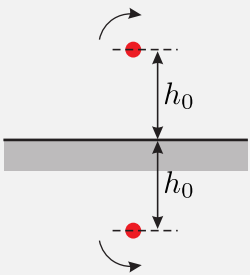


Illustration of the image method for the straight vortex filament near a flat wall

- Trajectory is plotted correctly 0.25
- Correct direction of velocity 0.1
- Correct expression for velocity magnitude 0.15

Consider a straight vortex placed in a corner at a distance h_0 from both walls.

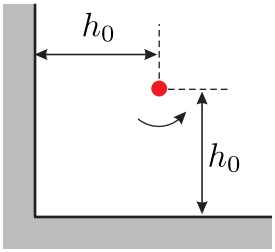


Figure 12: Straight vortex filament in a corner

E2 (0.75)

What is the initial velocity v_0 of the vortex?

The velocity of the filament is given by superposition of the velocities $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ induced by the image vortices 1, 2 and 3, respectively (see Fig. in E3 solution). One readily obtains

$$v_1 = \frac{\kappa}{2h_0}, \quad v_2 = \frac{\kappa}{2\sqrt{2}h_0}, \quad v_3 = \frac{\kappa}{2h_0}.$$

The modulus of the filament velocity at the initial moment is

$$v_0 = |\vec{v}_1 + \vec{v}_2 + \vec{v}_3| = \sqrt{2}v_1 - v_2 = \boxed{\frac{\kappa}{2\sqrt{2}h_0}}$$

- Ideas of using superposition principal and technique of image charges 0.25
- Correct direction of initial velocity 0.2
- Correct expression for initial velocity magnitude 0.3

E3 (0.5)

Draw the trajectory of the vortex.

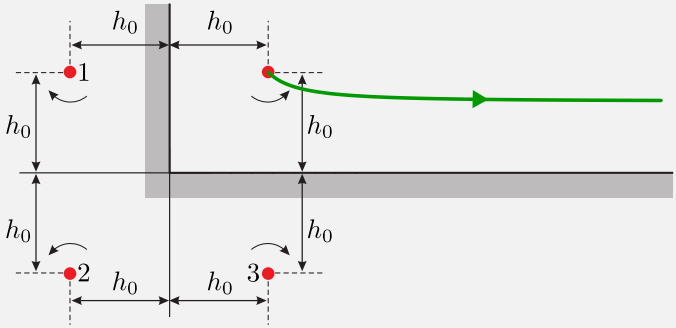


Image vortices in the corner.

- The trajectory has correct form and asymptote 0.5

E4 (1.0)

What is the velocity of the vortex v_∞ after very long time?

Energy for the system of vortices is proportional to

$$U_{\text{tot}} \propto \log \frac{\sqrt{x^2+y^2}}{a} - \log \frac{x}{a} - \log \frac{x}{a}. \tag{21}$$

The energy conservation implies that

$$C = \frac{x^2+y^2}{x^2y^2} = \frac{2}{h_0^2} \tag{22}$$

is constant along the trajectory. After very long time $y \to h_0/\sqrt{2}$ and the vortex velocity is

$$v_\infty = \frac{\kappa}{h_0\sqrt{2}}. \tag{23}$$

- Correct expression for velocity after very long time 1.0

F. Charges + Walls (1.0)

The vortices in this section are charged with uniform linear density $\lambda < 0$.

F1 (0.5)

Draw the trajectory of a straight vortex initially placed at a distance h_0 to a dielectric wall with charge density $\sigma < 0$. Find its velocity as a function of time.

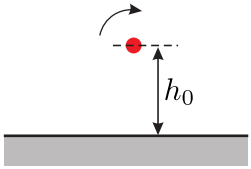
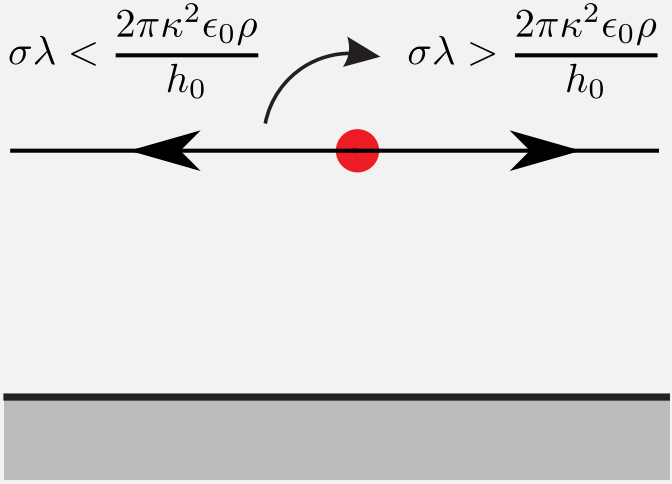


Figure 13: Straight vortex near a charged wall



The vortex velocity in electric field $E = \sigma/(2\varepsilon_0)$ is summed up with the fluid velocity generated by the image vortex

$$v = \frac{\kappa}{2h_0} - \frac{\sigma\lambda}{4\pi\varepsilon_0\kappa\rho} \tag{24}$$

and parallel to the wall. Its direction depends on the competition of the two terms above.

- Trajectory is plotted correctly 0.25
- Correct expression for velocity 0.25

F2 (0.5)

Draw the trajectory of a straight vortex initially placed at a distance h_0 to a conducting wall. Find its velocity as a function of time.

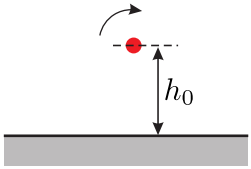
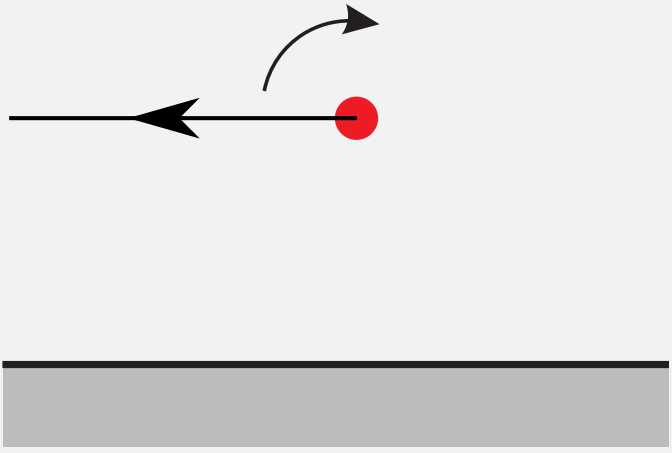


Figure 14: Straight vortex near a conducting wall



The conducting wall produces electric field identical to the field of an image filament with opposite charge density $E = -\lambda/(4\pi\varepsilon_0 h_0)$. The velocity is then

$$v = \frac{\lambda^2}{8\pi^2\varepsilon_0\kappa\rho h_0} + \frac{\kappa}{2h_0}. \tag{25}$$

- Trajectory is plotted correctly 0.25
- Correct expression for velocity 0.25

Evolution of Supermassive Black Holes Binary Solution

A. Dynamical Friction

A1. The deflection angle is defined from: $\tan\alpha \approx \alpha = \frac{p_y}{p_x}$, assuming that $\alpha \ll 1$. One can find $p_y = \int F_y\, dt$, and according to Newton's gravity law

$$F_y = \frac{GMm}{b^2}\cos^3\varphi$$

Changing the variable $dt = \frac{bd\varphi}{v\cos^2\varphi}$ we have

$$p_y = \frac{GM}{bv}\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi = \frac{2GMm}{bv}.$$

Here we assume that the body moves along the stright line, due to $\alpha \ll 1$, see Fig 1. So $\alpha = \frac{p_y}{p} = \frac{2GM}{bv^2} = \frac{2b_1}{b}$.

A2. During the transit of a massive body, star's energy remains constant: $p_x^2 + p_y^2 = \text{const}$. Hence

$$(p - \Delta p_x)^2 + p_y^2 = p^2.$$

We know that $p_y \ll p$, so the momentum change along the x-axis:

$$\Delta p_x = -\frac{p_y^2}{2p} = -\frac{\alpha^2}{2}p = -\frac{2G^2M^2m}{b^2v^3}.$$

A3. To calculate net force we might integrate over stars with different impact parameters. The number of stars' transits during the time Δt equals $\Delta N = 2\pi bvn\, db\, \Delta t$, so force, decelerating the object along the x-axis,

$$F_{DF} = \frac{1}{\Delta t}\int \Delta p_x\, dN = -4\pi G^2M^2\frac{nm}{v^2}\int_{b_{min}}^{b_{max}}\frac{db}{b} = -4\pi G^2M^2\frac{\rho}{v^2}\log\Lambda \tag{1}$$

The above formulas are true only for $b > b_1$, so the lower integration limit is $b_{min} = b_1$, and the upper limit is determined by the galaxy size $b_{max} = R$. So we have

$$F_{DF} = -4\pi G^2M^2\frac{\rho}{v^2}\log\Lambda \tag{2}$$

where $\Lambda = R/b_1$.

A4. We calculate: $b_1 = \frac{GM}{v^2} = 11$ pc, $\log\Lambda = 7.6$.

B. Gravitational slingshot

B1. From the second Newton's law

$$\frac{mv^2}{2} = \frac{GMm}{4a^2},$$

and we have for the orbital velocity $v_{bin} = \sqrt{\frac{GM}{4a}}$. The system energy is

$$E = E_{\text{kin}} + U = 2 \cdot \frac{Mv^2}{2} - \frac{GM^2}{2a} = -\frac{GM^2}{4a}.$$

B2. From angular momentum conservation law

$$b\sigma = r_m v_0,$$

express v_0. Write down the energy conservation law

$$\frac{\sigma^2}{2} = \frac{v_0^2}{2} - \frac{GM_2}{r_m}$$

and derive $b = r_m\sqrt{1 + \frac{2GM_2}{\sigma^2 r_m}}$.

B3. To estimate the time between collisions let us use an analogy with the gas. As known from the molecular kinetic theory, given that molecules have radii r, thermal velocities v, and the molecular concentration is n, time t between collisions of one molecule with the others can be estimated from the relation $\pi r^2 vtn = 1$. In our problem b_{max} stands in place of the molecule radius, therefore for estimation it can be written $\Delta t = \frac{1}{\sigma b_{max}^2 n}$.

Estimate the maximal impact parameter b_{max}, corresponding to the star collision with the binary system. The star should reach the distance of a to the binary system to collide. When the star is at large distances from the SBH binary, it interacts with it as with a point object of mass $M_2 = 2M$. From the results of B.2, assuming $r_m = a$, we obtain $b_{max} = a\sqrt{1 + \frac{4GM}{\sigma^2 a}}$. Taking into account that $\sigma \ll \frac{GM}{a}$, simplify: $b_{max} = \frac{2}{\sigma}\sqrt{GMa}$, so we have

$$\Delta t = \frac{m\sigma}{GM\rho a}.$$

B4. During the one act of gravitational slingshot, star energy increases at average by $\Delta E_{star} = \frac{mv_{bin}^2}{2} - \frac{m\sigma^2}{2}$.
So after the one collision the energy of the binary system decreases by the same magnitude. Taking into account that $\sigma \ll v_{bin}$, we derive $\Delta E = -\frac{m}{2}v_{bin}^2$.

Average binary system energy loss rate equals $\frac{dE}{dt} = \frac{\Delta E}{\Delta t} = -\frac{G^2M^2\rho}{4\sigma}$. In respect to $\frac{dE}{dt} = \frac{GM^2}{4a^2}$, orbit radius variation rate can be estimated as

$$\frac{da}{dt} = -\frac{G\rho a^2}{\sigma}. \tag{3}$$

B5. Obtained equation can be easily integrated

$$\frac{da}{a^2} = -\frac{G\rho}{\sigma}dt. \tag{4}$$

To reduce the radius twice it takes time $T_{SS} = \frac{\sigma}{G\rho a_1} = 0.0048\,\text{Gy}$.

C. Emission of gravitational waves

C1. Using previous results (B) one can obtain:

$$\frac{dE}{dt} = \frac{d}{dt}\left(-\frac{GM^2}{4a}\right) = \frac{GM^2}{4a^2}\frac{da}{dt}; \tag{5}$$

This leads to the differential equation:

$$\frac{GM^2}{4a^2}\frac{da}{dt} = -\frac{1024}{5}\cdot\frac{G\omega^6 M^2 a^4}{c^5}, \tag{6}$$

where angular velocity is known from part B: $\omega = \sqrt{\frac{GM}{4a^3}}$.
The desirable result is:

$$\frac{da}{dt} = -\frac{G^3M^3}{c^5a^3} \tag{7}$$

C2. Solving the equation one can obtain:

$$a^3\frac{da}{dt} = -\frac{G^3M^3}{c^5} \Rightarrow \frac{a_2^4}{4} \approx \frac{256}{5}\cdot\frac{G^3M^3}{c^5}\cdot T_{GW}; \tag{8}$$

The final result for time:

$$T_{GW} = \frac{5}{1024}\cdot\frac{a_2^4c^5}{G^3M^3}. \tag{9}$$

C3. Solving the previous equation:

$$a_H = \sqrt[4]{\frac{1024}{5}\cdot\frac{t_H G^3M^3}{c^5}} \approx 0.098\ pc. \tag{10}$$

D. Full evolution

D1. The galaxy is spherically symmetric, so mass enclosed within a sphere of radius r equals $m(r) = \int_0^r 4\pi x^2\rho(x)\,dx = \frac{\sigma^2 r}{G}$. Thus the free fall acceleration of the body equals in the gravitational field of stars is $g(r) = \frac{Gm(r)}{r^2} = \frac{\sigma^2}{r}$. Therefore the body velocity is determined by relation $\frac{v^2}{r} = g = \frac{\sigma^2}{r}$, which means $v = \sigma$ =const.

D2. The energy of SBH in this gravitational field is

$$E = \frac{M\sigma^2}{2} + U$$

and $\frac{dU}{da} = g(r)M = \frac{M\sigma^2}{a}$. So $\frac{dE}{dt} = \frac{dU}{dt} = \frac{dU}{da}\frac{da}{dt} = \frac{M\sigma^2}{a}\frac{da}{dt}$.
Using the result of A2 we have

$$\frac{dE}{dt} = -F_{Df}v = -4\pi G^2M^2\frac{\rho(r)}{v}\log\Lambda = -\frac{GM^2\sigma\log\Lambda}{a^2},$$

and the answer is

$$\frac{da}{dt} = -\frac{GM\log\Lambda}{a\sigma} \tag{11}$$

D3. To estimate one can assume that SBHs form a binary when the mass of stars inside the sphere of radius a equals to M:

$$m(a) = \frac{\sigma^2 a}{G} = M,$$

so $a_1 = \frac{GM}{\sigma^2} = 11$ pc.

D4. Integrating the equation (11) we have

$$\frac{a_0^2 - a_1^2}{2} = \frac{GM \log \Lambda}{\sigma} T_1$$

and using that $a_1 \ll a_0$ we have

$$T_1 = \frac{a_0^2 \sigma}{2GM \ln \Lambda} = 0.12\,\text{Gy}.$$

D5. Total energy losses are caused by gravitational slingshot and gravitational waves emission, so

$$\frac{dE}{dt} = \frac{G^2 M^2 \rho}{4\sigma} - \frac{64 G^4 M^5}{5c^5 a^5} \tag{12}$$

Energy losses due to GW dominates when $\frac{G^2 M^2 \rho_1}{4\sigma} < \frac{64 G^4 M^5}{5c^5 a^5}$ i.e. $a < a_2$ where

$$a_2^5 = \frac{256}{5} \frac{G^2 M^3 \sigma}{c^5 \rho_1}.$$

Numerical answers are $\rho_1 = 6 \times 10^3 M_S/\text{pc}^3$ and $a_2 = 0.026$ pc.

D6. For rough approximation it can be considered that at the slingshot stage losses are due to slingshot only, so T_2 is calculated analogiously to B5:

$$T_2 = \frac{\sigma}{G\rho a_2} = 0.27\,\text{Gy}$$

And at the GW emission stage losses are due to GW only, so T_3 is calculated directly from to C2:

$$T_3 = \frac{a_2^4 c^5}{16 G^3 M^3} = \frac{\sigma}{16 G \rho a_2} = 0.067\,\text{Gy}$$

D7. Total time of SBH binary evolution from the moment of galaxies merging to SBH merging equals

$$T_{evol} = T_1 + T_2 + T_{GW} = 0.12 + 0.27 + 0.07\,\text{Gy} = 0.46\,\text{Gy}$$

Space Debris

APhO 2017

Introduction

In more than half a century of space operations quite a large number of man-made objects have been amassed near Earth. The objects that do not serve any particular purpose are called *space debris*. The most attention is usually paid to the larger debris objects, i.e. defunct satellites and spent rocket upper stages, which stay in orbit after delivering their payload. Collisions of such objects with each other may result in thousands of fragments endangering all current space missions.

There is a well-known hypothetical scenario, according to which certain collisions may cause a cascade where each subsequent collision generates more space debris that increase the likelihood of new collisions. Such a chain reaction, resulting in the loss of all near-Earth satellites and making impossible further space programs, is called the *Kessler syndrome*.

To prevent such undesirable outcome special missions are planned to remove large debris object from their present orbits either by tugging them to the Earth's atmosphere or to graveyard orbits. To this end a specially designed spacecraft – a space tug – must capture a debris object. However, before capturing an uncontrolled object it is important to understand its rotational dynamics.

We suggest you to take part in planning of such a mission and find out how the rotational dynamics of a debris object changes in time under the influence of different factors.

Rocket Stage Schematic

The debris object to be considered is a "Kerbodyne 42" rocket upper stage, whose schematic is shown in Fig. 1. The circle line in Fig. 1 marks the outline of a spherical fuel tank.

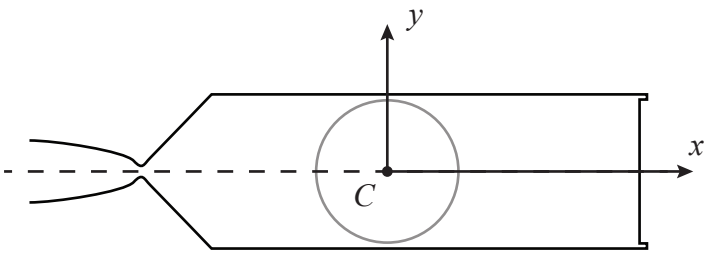


Figure 1: "Kerbodyne 42" upper stage

We introduce a body-fixed reference frame Cxy with the origin in the center of mass C, x being the symmetry axis of the stage, and y perpendicular to x. The inertia moments with respect to x and y axes are J_x and J_y ($J_x < J_y$).

A. Rotation

Consider an arbitrary initial rotation of the stage with angular momentum L (Fig. 2), where θ is the angle between the symmetry axis and the direction of angular momentum. Fuel tank

at this point is assumed to be empty. No forces or torques act upon the stage.

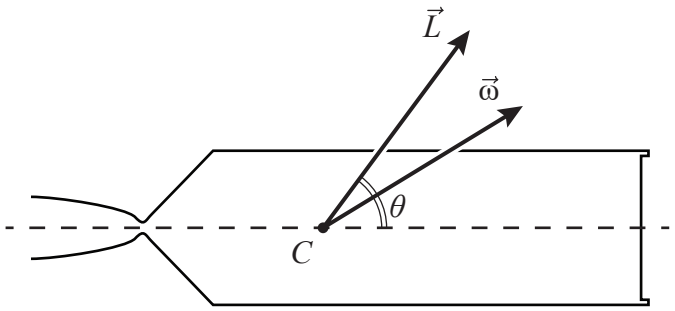


Figure 2: Rocket stage rotation

1.(0.2 pts) Find the projections ω_x and ω_y of angular velocity $\vec{\omega}$ on x and y, given that $\vec{L} = J_x\omega_x\vec{e_x} + J_y\omega_y\vec{e_y}$ for material symmetry axes x and y with unit vectors $\vec{e_x}$ and $\vec{e_y}$. Provide the answer in terms of $L = |\vec{L}|$, angle θ, and inertia moments J_x, J_y.

$$\omega_x = \frac{L\cos\theta}{J_x}, \tag{A1}$$

0.1 point

$$\omega_y = \frac{L\sin\theta}{J_y}. \tag{A2}$$

0.1 point

2.(0.4 pts) Find the rotational energy E_x associated with rotation ω_x and E_y associated with rotation ω_y. Find total rotational kinetic energy $E = E_x + E_y$ of the stage as a function of the angular momentum L and $\cos\theta$.

$$E_x = \frac{J_x\omega_x^2}{2}, \tag{A3}$$

0.1 points

$$E_y = \frac{J_y\omega_y^2}{2}, \tag{A4}$$

0.1 points

$$E(\theta) = E_x + E_y = \frac{J_x\omega_x^2}{2} + \frac{J_y\omega_y^2}{2} = \frac{L_x^2}{2J_x} + \frac{L_y^2}{2J_y} = \frac{L^2}{2J_y} + \frac{L^2}{2}\left(\frac{1}{J_x} - \frac{1}{J_y}\right)\cos^2\theta. \quad \text{(A5)}$$

0.2 points

In the following questions of Section A consider the stage's free rotation with the initial angular momentum L and $\theta(0) = \theta_0$.

3. (1.2 pts) Let us denote by x_0 the initial orientation of the stage's symmetry axis Cx with respect to the inertial reference frame. Using conservation laws find the maximum angle ψ, which the stages symmetry axis Cx makes with x_0 during the stage's free rotation.

Note: Since there are no external torques acting upon the stage, the angular momentum vector remains constant.

Both kinetic energy and angular momentum are conserved, and $\cos^2\theta$ can be obtained from equation A5.
Consequently, the set of values that θ can take is discrete (one value in each quadrant for every value of $\cos^2\theta$), and in the process of continuous motion θ cannot change its initial value. Therefore the stage's axis of symmetry moves around $\vec{L}$ making a conic surface with aperture $2\theta_0$. Consequently

$$\psi = 2\theta_0. \quad \text{(A6)}$$

1.2 points for the correct answer for ψ.
If the correct answer is not provided 1.0 point is given for the proof that $\theta(t) = \theta_0$ and does not change in time.
If this is not done:

- 0.2 points for the formula expressing the angular momentum conservation,
- 0.2 points for the formula expressing the energy conservation,
- 0.2 points for the formula expressing θ through any given constant parameters of the problem

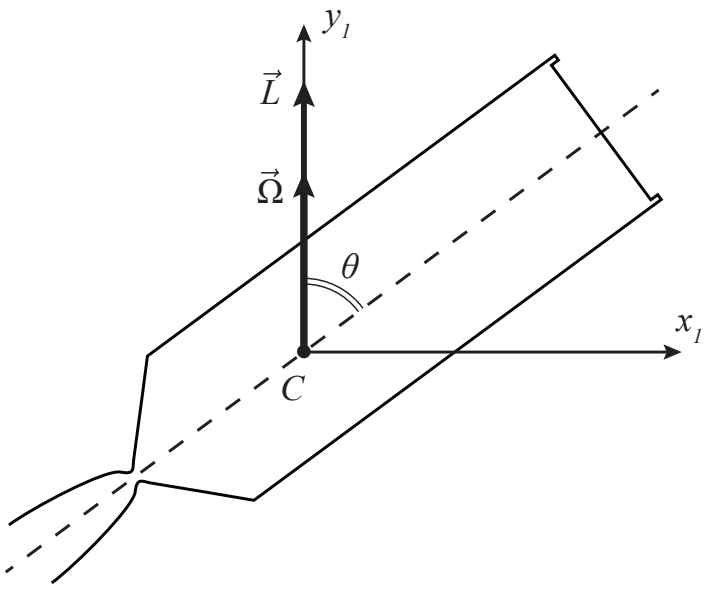


Figure 3:

Let us now introduce the reference frame $Cx_1y_1z_1$ with y_1 along the constant angular momentum vector $\vec{L}$ (Fig. 3). This reference frame rotates about y_1 in such a way, that the stage's symmetry axis always belongs to the Cx_1y_1 plane.

4. (2.0 pts) Given L, $\theta(0) = \theta_0$ and inertia moments J_x, J_y, find the angular velocity $\Omega(t)$ of the reference frame Cx_1y_1 about y_1 and direction (i.e. angle $\gamma_s(t)$ that $\vec{\omega}_s(t)$ makes with the symmetry axis Cx) and absolute value of angular velocity of the stage $\vec{\omega}_s(t)$ relative to the reference frame Cx_1y_1 as functions of time.

Note: angular velocity vectors are additive $\vec{\omega} = \vec{\omega}_x + \vec{\omega}_y = \vec{\Omega} + \vec{\omega}_s$.

The symmetry axis is at rest with respect to the rotating reference frame, because $\theta(t) = \theta_0$ and the symmetry axis always belongs to the Cx_1y_1 plane. Hence, $\vec{\omega}_s$ must be collinear to the symmetry axis at all times. Thus

$$\gamma_s(t) = 0.$$

0.5 points

Projecting the sum $\vec{\Omega} + \vec{\omega}_s$ onto Cx and Cy yields for any t:

$$\omega_s + \Omega\cos\theta = \omega_x = \frac{L\cos\theta}{J_x}, \tag{A7}$$

$$\Omega\sin\theta = \omega_y = \frac{L\sin\theta}{J_y}. \tag{A8}$$

0.25 points for each of the equations A7 and A8

Whence

$$\Omega = \frac{L}{J_y}. \tag{A9}$$

Thus Ω does not depend on time.

1.0 points

Taking into account that $\theta(t) = \theta_0$:

$$\omega_s = \left(\frac{1}{J_x} - \frac{1}{J_y}\right) L \cos\theta_0. \tag{A10}$$

And ω_s also does not depend on time.

0.5 points

NB:
1.0 points is given for the correct answer for Ω.
For ω_s 0.5 points is given for the direction of $\vec{\omega_s}$ (along Cx)
0.5 points is given for A10

Alternatively:
0.25 points is given for any of the A7, A8 equations

B. Transient Process

Most of the propellant is used during the ascent, however, after the payload has been separated from the stage, there still remains some fuel in its tank. Mass m of residual fuel is negligible in comparison to the stage's mass M. Sloshing of the liquid fuel and viscous friction forces in the fuel tank result in energy losses, and after a transient process of irregular dynamics the energy reaches its minimum.

1.(0.6 pts) Find the value θ_2 of angle θ after the transient process, for arbitrary initial values of L and $\theta(0) = \theta_1 \in (0, \pi/2)$.

Interaction of the residual fuel with the fuel tank walls can be considered an internal force. Hence, as before, no external forces or torques act upon the system, and the angular momentum is conserved.
For the given initial value of θ and knowing that $J_x < J_y$, it is easily shown from A5 that $E(\cos\theta)$ reaches its minimum for $\theta = \pi/2$.
Thus

$$\theta_2 = \frac{\pi}{2}. \tag{B1}$$

0.6 points

2.(0.6 pts) Calculate the value ω_2 of angular velocity ω after the transient process, given that initial angular velocity $\omega(0) = \omega_1 = 1\ rad/s$ makes an angle of $\gamma(0) = \gamma_1 = 30^\circ$ with the stage's symmetry axis. The moments of inertia are $J_x = 4200\ kg \cdot m^2$ and $J_y = 15\ 000\ kg \cdot m^2$.

B1 implies that after the transient process the stage rotates about the axis perpendicular to its symmetry axis.

0.2 points

Final angular velocity value ω_2 can be obtained from the angular momentum conservation law:

$$\omega_2 = \frac{L}{J_y} = \frac{\sqrt{J_x^2 \cos^2 \gamma_1 + J_y^2 \sin^2 \gamma_1}}{J_y} \omega_1 \tag{B2}$$

0.2 points

$$\omega_2 \approx 0.56\ rad/s. \tag{B3}$$

0.2 points

C. Magnetic Field

Another important factor in rotational dynamics of a debris rocket stage, which is orbiting the Earth, is its interaction with the Earth's magnetic field. Let us first consider an auxiliary problem.

Torque due to Eddy Currents

Let us place a thin-walled nonmagnetic spherical shell with wall thickness D and radius R in a uniform magnetic field $\vec{B}$, which slowly changes so that its derivative $\dot{\vec{B}}$ is a constant vector making angle α with the direction of $\vec{B}$ (Fig. 4). Electrical resistivity of the shell's material is ρ.

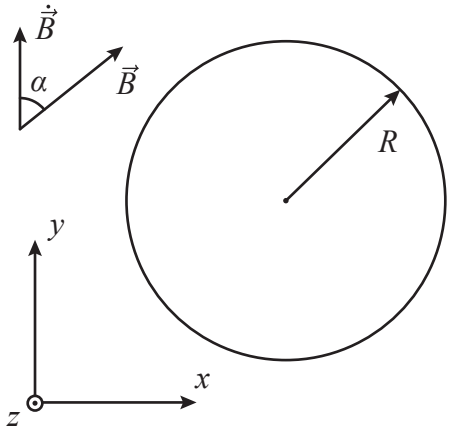


Figure 4: Spherical shell in magnetic field

1. (1.0 pts) Find the induced magnetic moment $\vec{\mu}$ of the shell, neglecting its self-inductance. Provide the answer for $\vec{\mu}$ in the form of projections on xyz (see Fig. 4).

Let us cut the sphere into ring slices so that $\dot{\vec{B}}$ is perpendicular to their planes and introduce angle φ as shown in Fig. 5.

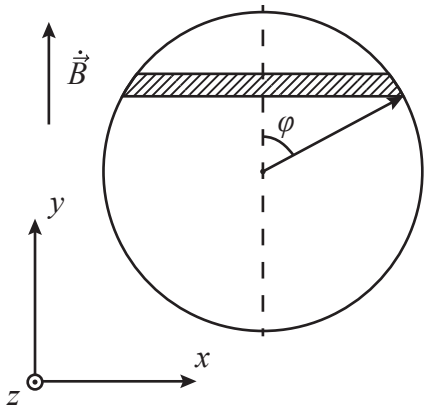


Figure 5:

According to Faraday's law the absolute value of eddy current EMF, induced in such a slice by the varying magnetic field is

$$\mathcal{E} = \dot{\Phi} = S\dot{B} = \pi R^2 \sin^2\varphi \dot{B} \tag{C1}$$

0.2 points

The ring slice resistance is

$$dr = \frac{2\pi\rho R \sin\varphi}{DRd\varphi}. \tag{C2}$$

0.2 points

Current in the ring slice

$$dI = \mathcal{E}/dr = \frac{1}{2\rho} DR^2 \dot{B} \sin\varphi d\varphi \tag{C3}$$

0.1 points

And, finally, magnetic moment:

$$d\mu = SdI = \frac{\pi}{2\rho} DR^4 \dot{B} \sin^3\varphi d\varphi = \frac{1}{4\rho} \dot{B} dJ, \tag{C4}$$

where dJ is the moment of inertia for a slice ring of unit density with respect to the central axis, which is parallel to y.

0.2 points

Thus

$$\mu = \frac{1}{4\rho} J\dot{B} = \frac{2\pi}{3\rho} DR^4 \dot{B}, \tag{C5}$$

where J is the moment of inertia of the sphere with respect to the axis, passing through its center. Taking into account the direction:

$$\mu_x = 0, \tag{C6}$$

$$\mu_y = -\frac{2\pi}{3\rho} DR^4 \dot{B}, \tag{C7}$$

$$\mu_z = 0. \tag{C8}$$

0.1 points for each $\vec{\mu}$ component

2. (0.3 pts) Find the torque $\vec{M}$ acting on the spherical shell. Provide the answer for $\vec{M}$ in the form of projections on xyz (see Fig. 4).

The torque is given by $\vec{M} = [\vec{\mu}, \vec{B}]$. It is directed along the z axis and equals

$$M_z = \mu B \sin\alpha = \frac{2\pi}{3\rho} D R^4 B \dot{B} \sin\alpha. \tag{C9}$$

0.1 points for each $\vec{M}$ component

NB: Alternatively, if the task of the previous assignment (find $\vec{\mu}$) is not completed, but the answer for $\vec{M}$ is, nevertheless, provided, the points for intermediate steps from the previous assignment (except 0.3 points for $\vec{\mu}$ components) are redistributed for the actions to find $\vec{M}$.

Attitude Motion Evolution in the Earth's Magnetic Field

Let us find out how the rotation changes for a rocket stage, which moves in a circular polar orbit with orbital period $T = 100\ min$ (Fig. 6). It transpires that the characteristic times of dynamics due to interaction with the geomagnetic field are much greater than the duration of the transient process. We will now study what happens to the rocket stage after the transient process has completed. To start our analysis consider the stage rotating with angular velocity ω_2 about the axis perpendicular to the orbital plane.

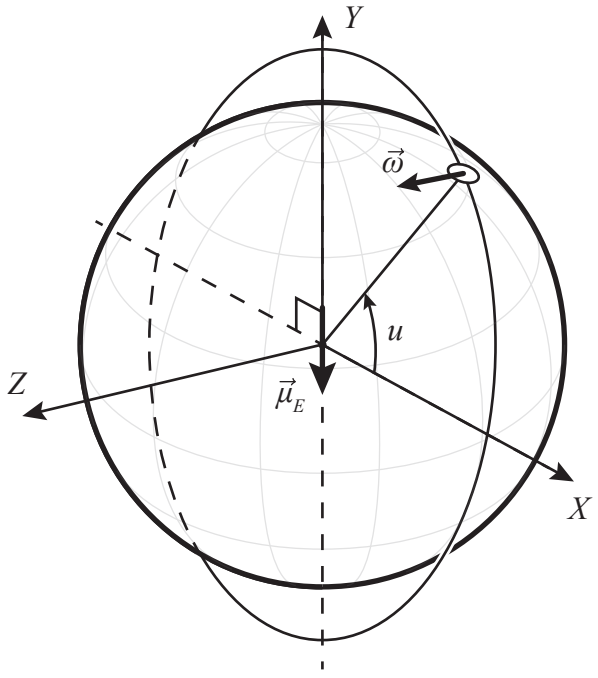


Figure 6: The orbit

1.(0.4 pts) The Earth's magnetic field $\vec{B}_E$ can be modeled as the magnetic field of a point dipole in the Earth's center. Its dipole moment $\vec{\mu}_E$ is directed opposite to Y axis. The absolute value of the Earth's magnetic field B at the point where the orbit crosses the equatorial plane

XZ is $B_0 = 20\ \mu T$. Find $\vec{B}_E(u)$ at a current position of the stage in the orbit defined by the angle u as shown in Fig. 6. The positive direction of u is along with the orbital motion. Provide the answer in the form of the projections of $\vec{B}_E(u)$ on XYZ axes.

Note: Magnetic field of a dipole at point $\vec{r}$ is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3\,(\vec{\mu}\cdot\vec{r})\,\vec{r}}{r^5} - \frac{\vec{\mu}}{r^3}\right).$$

Note: It may facilitate subsequent calculations if projections of $\vec{B}_E(u)$ are given as functions of $2u$ instead of u.

Let R_O be the orbit radius. The dipole field formula at point $\vec{r} = (R_O\cos u, R_O\sin u, 0)$ and $\vec{\mu} = (0, -\vec{\mu}_E, 0)$ yield

$$B_X = -\frac{3}{2}\frac{\mu_0\mu_E}{4\pi R_O^3}\sin 2u, \tag{C10}$$

$$B_Y = \left(1 - 3\sin^2 u\right)\frac{\mu_0\mu_E}{4\pi R_O^3}, \tag{C11}$$

$$B_Z = 0. \tag{C12}$$

0.05 for each $\vec{B}$ component, if no final answer (see below) is obtained.

At point, where the orbit passes through the equatorial plane ($u = 0$) the magnetic field is

$$B_X = 0, \tag{C13}$$

$$B_Y = \frac{\mu_0\mu_E}{4\pi R_O^3}, \tag{C14}$$

$$B_Z = 0. \tag{C15}$$

Thus $B_0 = \frac{\mu_0\mu_E}{4\pi R_O^3}$.

0.1 points for B_0

Finally, the Earth's magnetic field is:

$$B_X(u) = -\frac{3}{2}B_0\sin 2u, \tag{C16}$$

$$B_Y(u) = \frac{1}{2}\left(3\cos(2u) - 1\right)B_0, \tag{C17}$$

$$B_Z(u) = 0. \tag{C18}$$

0.1 points for each component of $\vec{B}$

The "Kerbodyne 42" rocket upper stage is mostly made of wood, and the only conductive

material is used for its cryogenic fuel tank. We, therefore, consider the stage's interaction with the geomagnetic field as that of the spherical shell with wall thickness $D = 2\ mm$, radius $R = 4\ m$ and resistivity $\rho = 2.7 \cdot 10^{-8}\ \Omega \cdot m$.

2.(**1.3 pts**) Find the torque $\vec{M}(u)$ acting on the stage, as it rotates with angular velocity ω collinear to Z. Provide the answer for $\vec{M}(u)$ in the form of projections on XYZ.

Using C9 requires us to find the magnetic field derivative in the body frame.
Consider a body frame xyz, whose axis z is collinear to Z and plane xy is rotated by angle β with respect to XY. Magnetic field in this reference frame is

$$B_x = B_X \cos\beta + B_Y \sin\beta, \tag{C19}$$
$$B_y = -B_X \sin\beta + B_Y \cos\beta, \tag{C20}$$
$$B_z = B_Z = 0. \tag{C21}$$

0.1 point for any idea that provides understanding that there are two processes in which B changes with respect to body-frames – orbital motion and rotational dynamics. The same 0.1 point is given for any approach overcoming this issue.

The derivative of magnetic field is therefore

$$\begin{aligned}
\dot{B}_x =& \dot{B}_X \cos\beta + \dot{B}_Y \sin\beta + (-B_X \sin\beta + B_Y \cos\beta)\,\dot{\beta} = \\
=& (B'_X(u)\cos\beta + B'_Y(u)\sin\beta)\,\dot{u} + (-Bh_X \sin\beta + B_Y \cos\beta)\,\dot{\beta}, \\
\dot{B}_y =& -\dot{B}_X \sin\beta + \dot{B}_Y \cos\beta + (-B_X \cos\beta - B_Y \sin\beta)\,\dot{\beta} = \\
=& (-B'_X(u)\sin\beta + B'_Y(u)\cos\beta)\,\dot{u} + (-B_X \cos\beta - B_Y \sin\beta)\,\dot{\beta}, \\
\dot{B}_z =& 0.
\end{aligned}$$

0.1 points for each component of $\dot{\vec{B}}$ related to the orbital motion
0.1 points for each component of $\dot{\vec{B}}$ related to the rotational dynamics
Full points are also given if $\dot{\vec{B}}$ is found in the vector form

Substituting $\dot{u} = 2\pi/T$ and $\dot{\beta} = \omega$ and using the expressions C9, C16, and C17, we obtain that the torque is directed along z and equals

$$M_z = \frac{2\pi}{3\rho} D B_0^2 R^4 \left(\frac{3\pi}{T}\,(3 - \cos 2u) - \frac{\omega}{2}\,(5 - 3\cos 2u) \right) = M_Z. \tag{C22}$$

0.1 points for M_x and M_y,
0.5 points for M_z (for complicated calculations)

3. (**1.0**)Find the absolute value of angular velocity $\omega(t)$ as a function of time, given that the change in the stage's angular velocity over one orbital period is negligibly small.

We will average M_Z over u and use the obtained expression instead of M_Z. This helps getting rid of the members, containing $\cos 2u$:

$$\langle M_Z \rangle = \frac{2\pi}{3\rho} D B_0^2 R^4 \left(\frac{9\pi}{T} - \frac{5\omega}{2} \right). \tag{C23}$$

0.25 points for the explicit idea to average M_Z over u
0.25 for the correct expression for $\langle M_Z \rangle$

As the torque is directed along with the rotation axis, it does not change the axis' direction, which means that the obtained formula always holds for the rocket stage rotational dynamics. As we consider the transient process to have completed. It follows from B1 that the rocket stage rotates about the axis, which is perpendicular to its symmetry axis. Thus the angular momentum of the stage is

$$L_Z = J_y \omega. \tag{C24}$$

0.25 for the correct equation with the correct inertia moment

As $\dot{L}_Z = M_Z$ is the governing equation for the angular velocity:

$$\dot{\omega} = \frac{2\pi}{3 J_y \rho} D B_0^2 R^4 \left(\frac{9\pi}{T} - \frac{5\omega}{2} \right). \tag{C25}$$

Its solution is:

$$\omega(t) = \frac{18\pi}{5T} + \left(\omega_2 - \frac{18\pi}{5T} \right) e^{-\delta t}, \tag{C26}$$

where $\delta = \frac{5\pi}{3 J_y \rho} D B_0^2 R^4$.

0.25 points for the correct solution of the differential equation for ω
Alternatively 0.15 for the exponential dependence of ω from t.

4. (**1.0**) Find the ratio of the orbital period T and the rocket stage's rotation period T_s in the steady-state regime, which sets in after a long time.

From C26 it follows that the angular velocity asymptotically tends to $18\pi/2T$. Thus the ratio of the two periods

$$\frac{T}{T_s(\infty)} = \frac{T\omega(\infty)}{2\pi} = 9/5 = 1.8. \tag{C27}$$

1.0 for the correct result.

Part A

A1. According to diffraction grating formula

$h \sin \varphi = m\lambda$, where $m = 1$, thus

$$h \sin \varphi = \lambda \quad (1s)$$

A2.

φ, ^{o}	θ, ^{o}
35	61,5
36	55,5
37	49,5
38	45
39	39
40	35
41	31,5
42	25
43	18,5

A3. Using Bragg-Snell law (2) from the task:

$$2D\sqrt{n^2 - \sin^2 \theta} = m\lambda \quad (2s)$$

and (1s) we get

$$2D\sqrt{n^2 - \sin^2 \theta} = h \sin \varphi$$

$$n^2 - \sin^2 \theta = \left(\frac{h}{2D}\right)^2 \sin^2 \varphi$$

$$\sin^2 \theta = n^2 - \left(\frac{h}{2D}\right)^2 \sin^2 \varphi \quad (3s)$$

$$n = \sqrt{Intercept}, \quad (4s)$$

where intercept is taken from Fig.1

$$D = \frac{h}{2\sqrt{|Slope|}}, \quad (5s)$$

where slope is taken from Fig.1

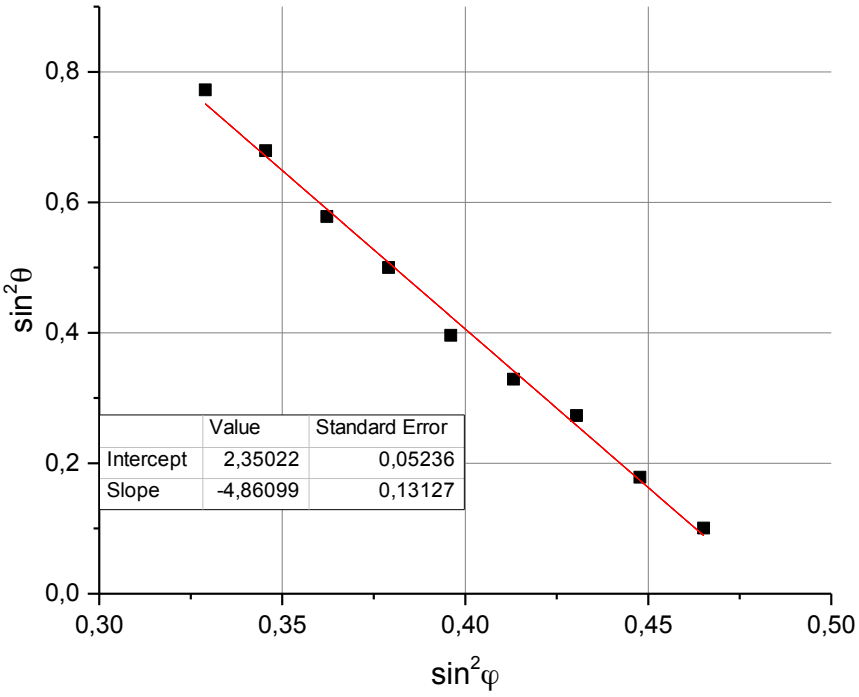


Fig. 1

A4. Taking slope and interception point from the Fig. 1 from (4s), (5s) we get:

$$n_X = 1.53$$

$$D_X = 227\ nm$$

Part B

B1. Minimum can be observed only in the red zone, so we use red laser with wavelength $\lambda = 650\ nm$

B2-B3.

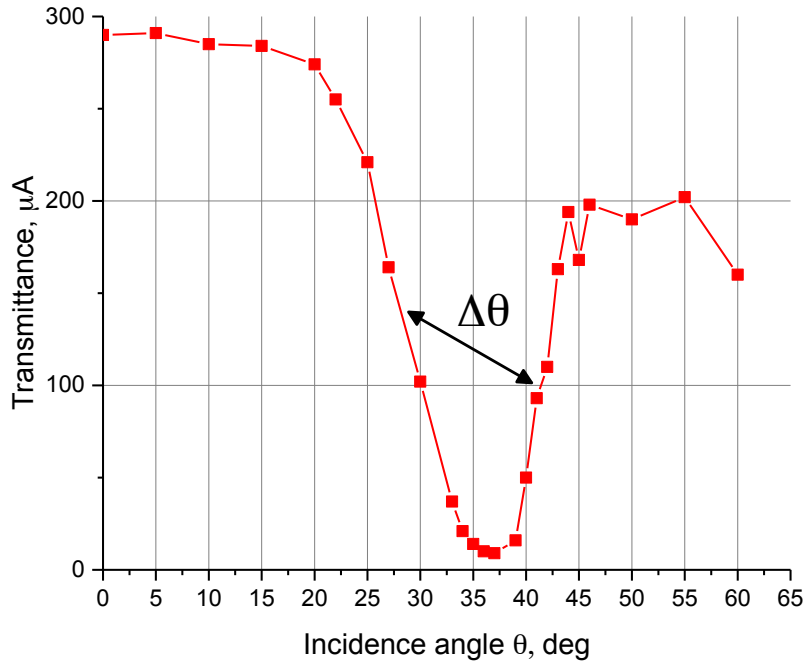


Fig. 2

B4. From the Fig. 2 we find minimum at $\theta_1 = 37^o$ with width of $\Delta\theta_1 = 14^o$

B5. Using (2s) we get

$$2D\sqrt{n^2 - \sin^2\theta_1} = m\lambda;$$

For the normal wavelength we can write

$$2Dn = m\lambda_X;$$

$$\lambda_X = \frac{\lambda n}{\sqrt{n^2 - \sin^2\theta_1}} = 707\ nm. \qquad (6s)$$

B6. Using (6s) we determine

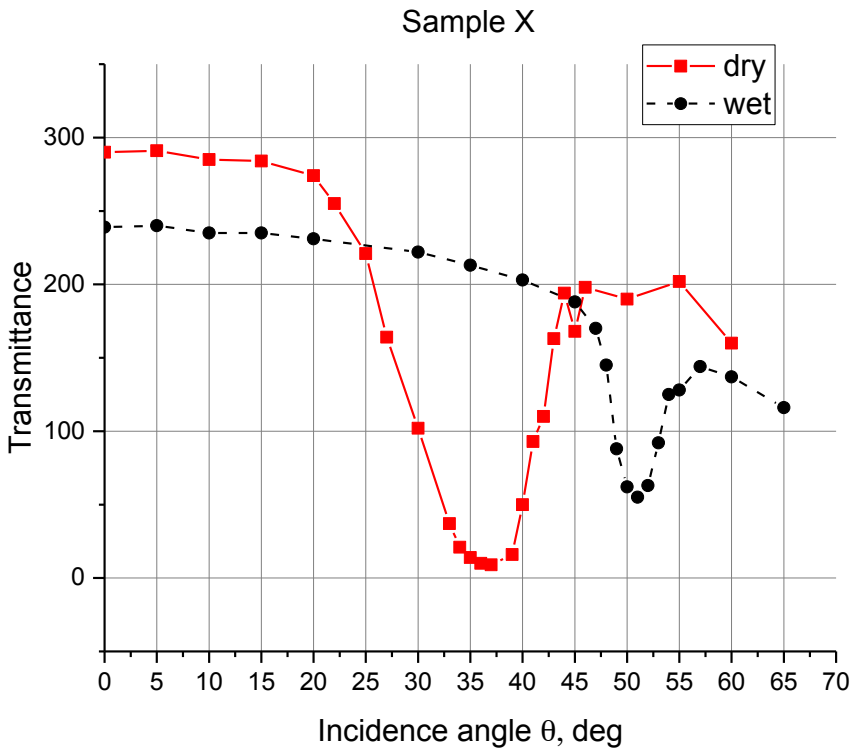
$\theta_{min} = 27^o \Rightarrow \lambda_{min} = 683\ nm$

$\theta_{max} = 41^o \Rightarrow \lambda_{max} = 719\ nm$

$\Delta\lambda = \lambda_{max} - \lambda_{min} = 36\ nm$

$$\Delta n = \frac{\pi}{2} n \frac{\Delta\lambda}{\lambda} = 0.12$$

B7. Minimum for the wet sample is at $\theta_2 = 51^{\circ}$.



2

B8. Using (2s) for n_{wet} and n_{dry} one can obtain

$$2D\sqrt{n_{dry}^2 - \sin^2\theta_1} = \lambda$$

$$2D\sqrt{n_{wet}^2 - \sin^2\theta_2} = \lambda$$

Using (4) and (5) from the task we determine

$$\frac{7}{9}p = n_{wet}^2 - n_{dry}^2 = \sin^2\theta_2 - \sin^2\theta_1$$

$$p = 0.31$$

$$n_{AAO} = \sqrt{\frac{n^2 - p}{! - p}} = 1.72$$

B9. Taking $\Delta n = 0.12$ and $p = 0.31$ we get

$$n_1 = n + \frac{\Delta n}{2} = 1.59$$

$$n_2 = n - \frac{\Delta n}{2} = 1.47$$

Equation (4) yields

$$p_1 = \frac{n_{AAO}^2 - n_1^2}{n_{AAO}^2 - 1} = 0.22$$

$$p_2 = \frac{n_{AAO}^2 - n_2^2}{n_{AAO}^2 - 1} = 0.41$$

Part C

C1. For the normal incidence ($\theta = 0$) we can observe three minima at angles $\varphi_1 = 43°, \varphi_1 = 35°, \varphi_1 = 29°$, therefore $\lambda_1^{sp} = 682\ nm, \lambda_2^{sp} = 574\ nm, \lambda_3^{sp} = 485\ nm$.

C2.

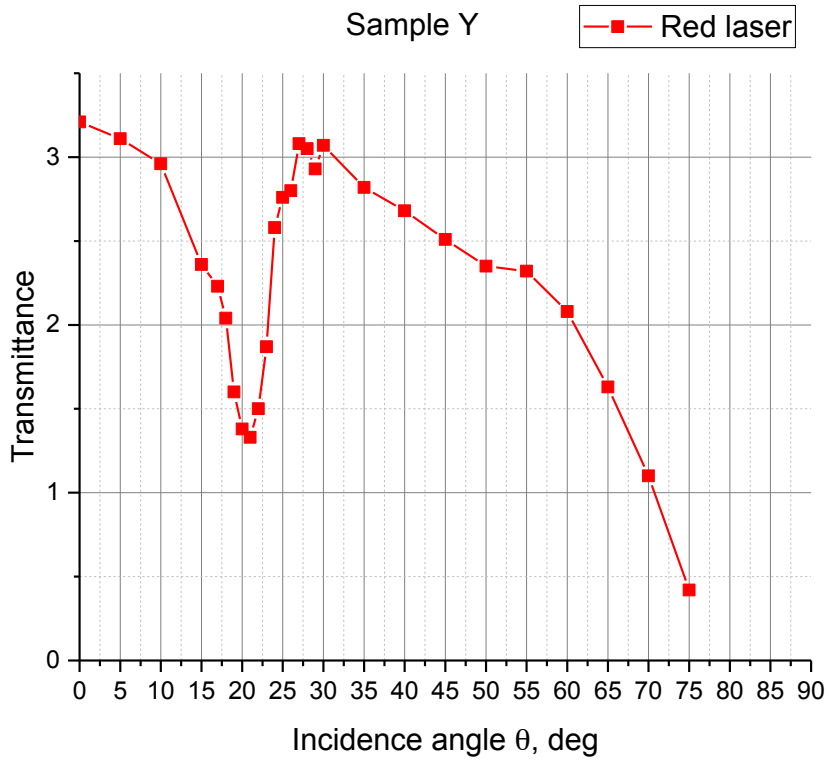


Fig. 4

C3.

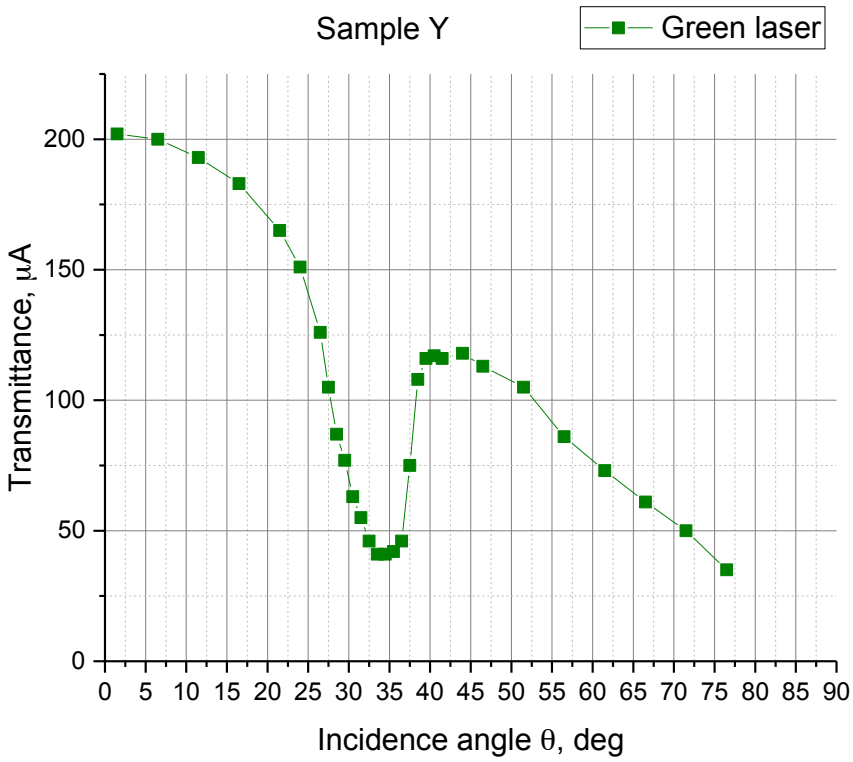

Sample Y
Green laser
Transmittance, μA
Incidence angle θ, deg
0
50
100
150
200
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90

Fig. 5

C4.

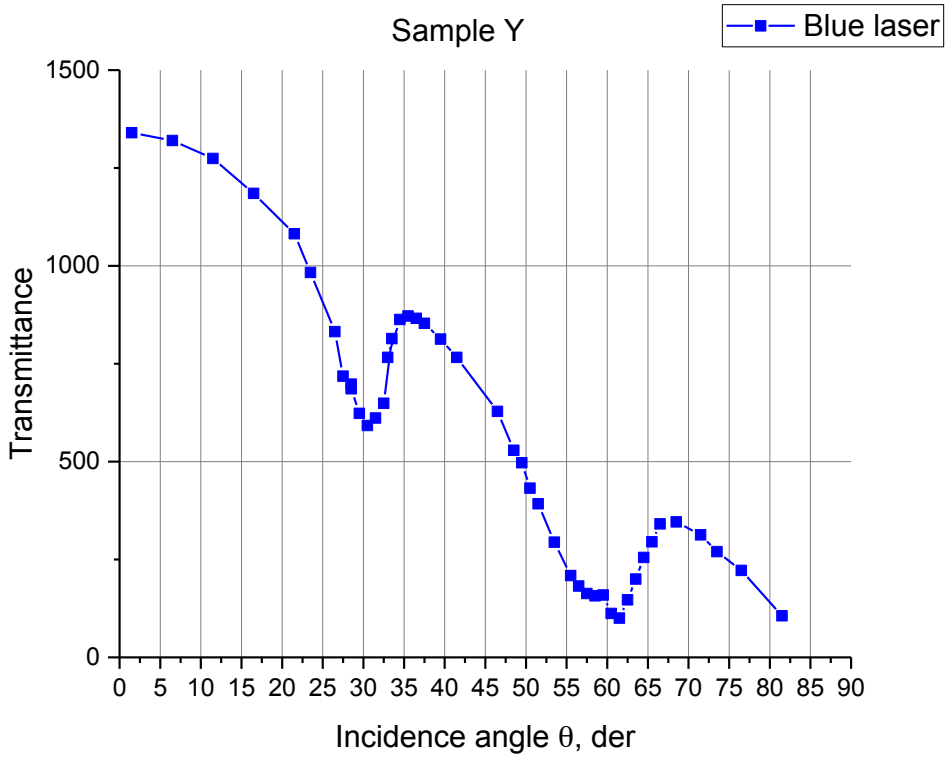

Sample Y
Blue laser
Transmittance
Incidence angle θ, der
0
500
1000
1500
0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90

Fig. 6

C5. Using (6s), we calculate normal wavelengths for an arbitrary order $m' = m + const$

$$\lambda_Y = \frac{\lambda n}{\sqrt{n^2 - \sin^2\theta_1}}$$

λ, nm	$\theta,\ deg$	$\lambda_Y,\ nm$	m'
659	21	678	1
530	34	569	2
400	57	478	3
400	30	423	4

C6. According to (2s) dependence $m'\left(\frac{1}{\lambda}\right)$ should be linear:

$$2Dn = (m' - const)\lambda$$

$$m' = \frac{2Dn}{\lambda} + const$$

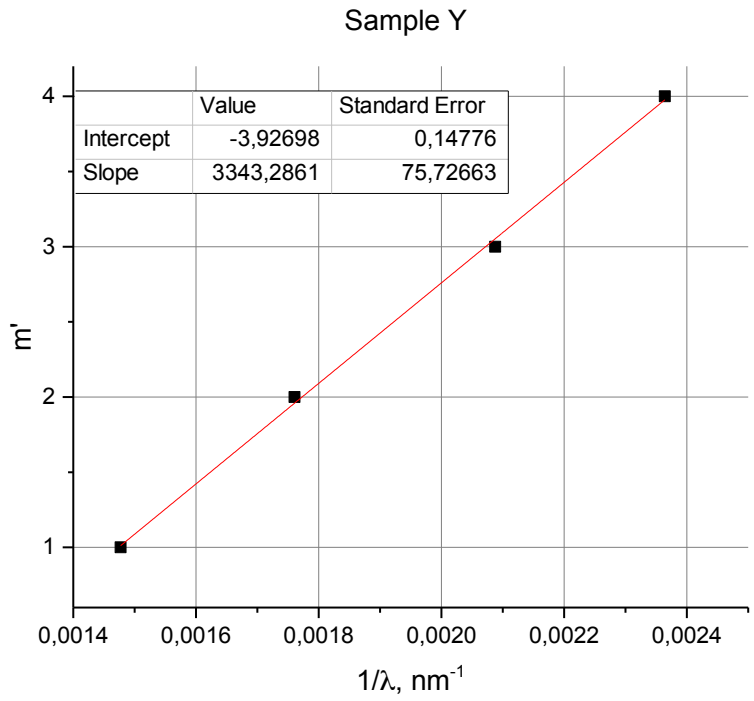


Fig. 7

$$m = m' - Intercept,$$

where intercept is taken from Fig.7

$$Intercept \simeq -4$$

$\lambda,\ nm$	m

677	5
568	6
479	7
423	8

C7.

$$D_Y = \frac{Slope}{2n}$$

where slope is taken from Fig. 7

$$D_Y = 1080\ nm$$

C8. Let I_1 be the half-sum of intensities to the left and to the right from the minimum, and I_2 be intensity in the minimum. Transmittance $t = \frac{I_2}{I_1}$.

$\lambda,\ nm$	t
677	0.42
568	0.29
479	0.22
423	0.63

Part D

D1. One can find up to 6 maximums:

$\lambda_Z,\ nm$
808
696
611
499
462
402

D2. According to (2s) dependence $m'\left(\frac{1}{\lambda}\right)$ should be linear:

$$2Dn = (m' - const)\lambda$$

$$m' = \frac{2Dn}{\lambda} + const$$

Dependence $m'\left(\frac{1}{\lambda}\right)$ will be linear if we take m'=1,2,3,5,6,8 for visible minimums (see fig.8).

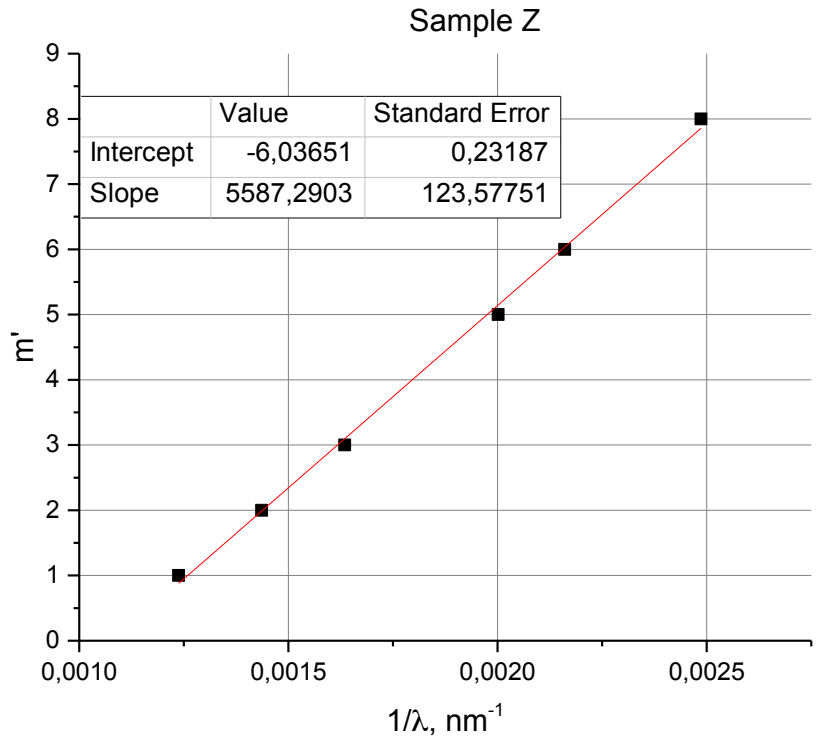


Fig. 8

$$m = m' - Intercept$$

From fig. 8 $Intercept = -6$.

$$m = m' + 6$$

λ_Z, nm	m'	m
808	1	7
696	2	8
611	3	9
499	5	11
462	6	12
402	8	14

D3.

$$D_Z = \frac{Slope}{2n},$$

where slope is taken from Fig. 8

$$D_Z = 1802\ nm$$

D4.

$$m = \frac{Slope}{\lambda} \Rightarrow \lambda = \frac{Slope}{m}$$

Missed minimums correspond to order m=10 and 13:

m	$\lambda_Z\ nm$
10	559
13	430

Part E

E1. We recognize the sample Y by such feature, that 2 central transmittance minimums are deeper than 2 side transmittance minimums. So, it is ***n-6***.

E2. For the sample Z $m = 10$ and $m = 13$ are missing, $m = 8, 9, 11, 12$ are not. So, it is ***hi5-5***.

Marking scheme
(minimal score 0.1pt)

Marker _____ Student _____

TOTAL _____

Task	Criteria	Max. points	Marker	Consensus
A1	Diffraction grating equation	0.1		
A2	$\theta(\varphi)$ measurements: - 0.2 for each point (not greater than 1.0) *φ in range 35-45^o θ in range 15-65^o*	1.0		
A3	Linearization: - $\sin^2\theta$ from $\sin^2\varphi$ dependence; Calculation of the $\sin^2\theta$ and $\sin^2\varphi$ values; Graph plotting *(only in linearized coordinates)* (axis labeled and scaled, experimental points plotted, linear fit line shown);	0.3 0.3 0.9= *0.3x3*		
A4	Sample parameters calculation *(scored only if A2 score ≥ 0.4):* Numeric parameters of the linear fit written All points approximation *Two points only* Formulae for D_X, n_X calculation using linear fit parameters n_X value *in range* ***1.4-1.6*** D_X value *in range* ***210-250 nm***	0.2 *(0.1)* 0.2 0.2 0.3		
	Part A total	**3.5**		
B1	$\lambda = 659nm$ selected	0.1		
B2	$I(\theta)$ measurements: Scored only if minimum is found in range *20-45^o* ***<u>and its transmittance value is lower that 0.5</u>*** - number of points 10 and more (*from 5 to 9, less than 5*) ; - angles range *(not less than 25° near minimum)*	0.6 (0.3; 0) 0.4		
B3	Graph plotted: (axis labeled and scaled, experimental points plotted, approximating curve shown);	0.2 0.6 0.2		
B4	θ_1 value in range *20-45^o* $\Delta\theta$ value in range *10-20^o*	0.1 0.1		
B5	Formula for λ_X Numerical value for λ_X in range $\pm$ **15 nm from the real value for the sample**	0.1 0.1		

B6	$\Delta\lambda$ calculation: Transition from θ to λ ; *Low $\Delta\theta$ value approximation* Numerical value for Δn in range **0,09 – 0,15**	 0.2 *0.1* 0.4		
B7	Numerical value for θ_2 in range *30-60°*	0.3		
B8	Formulae for p and n_{AAO} calculation Numerical value for p in range **0,15 – 0,35** Numerical value for n_{AAO} in range **1,55 – 1,75**	0.2+0.2 0.3 0.3		
B9	Formula for p_i versus n_i n_1, n_2 values selection Numerical values for: p_1 in range **0,05 – 0,25** p_2 in range **0,2 – 0,45**	0.1 0.1 0.2 0.2		
	Part B total	**5.0**		
C1	Values λ_1^{sp} in range $\pm$ **30 nm from the real value for the sample** λ_2^{sp} in range $\pm$ **30 nm from the real value for the sample** λ_3^{sp} in range $\pm$ **30 nm from the real value for the sample**	 0.2 0.2 0.2		
C2	$I(\theta)$ measurements for red laser: Scored only if minimum is found in range *10-30°* ***and its transmittance value is lower that 0.9*** - number of points is 10 and more (*from 5 to 9, less than 5*) ; - angle range *(from* 10° *to* 50°*)*	 0.3 *(0.2, 0)* 0.2		
C3	$I(\theta)$ measurements for green laser: Scored only if minimum is found in range *20-45°* ***and its transmittance value is lower that 0.5*** - number of points is 10 and more (*from 5 to 9, less than 5*) ; - angle range *(from* 10° *to* 50°*)*	 0.3 *(0.2, 0)* 0.2		
C4	$I(\theta)$ measurements for blue laser: Scored only if at least 1 minimum is found in range 10° - 65° ***and its transmittance value is lower that 0.9.*** - number of points is 10 and more (*from 5 to 9, less than 5*) ; - angle range *(from* 10° *to* 65°*)*	 0.3 *(0.2, 0)* 0.2		

C5	Normal wavelengths calculation (in the wavelength descending order) $\lambda_1^{(n)}$ value in range $\pm$**15 nm from the real value for the sample** $\lambda_2^{(n)}$ value in range $\pm$**15 nm from the real value for the sample** $\lambda_3^{(n)}$ value in range $\pm$**15 nm from the real value for the sample** $\lambda_4^{(n)}$ value in range $\pm$**15 nm from the real value for the sample**	0.1 0.1 0.1 0.2		
C6	Appropriate method Correct value for m corresponding to $\lambda_i^{(n)}$ (*in case of error in* m ± 1*)*	0.2 4x0.2 *(0.1)*		
C7	D_Y value in range **1000 – 1160 nm**	0.2		
C8	I_1, I_2 values obtained for 3-4 minima (*for 1-2 minima)* t numerical values: t_1 >0.45 t_2 <0.35 t_3 <0.35 t_4 >0.45	0.2 *(0.1)* 0.1 0.1 0.1 0.1		
	Part C total	**4.5**		
D1	Experimental setup for wavelength measurement sketch; Formula for $\lambda^{(n)}$ calculation; Wavelengths values (in the descending order) $\lambda_1^{(n)}$ value in range 790 -840 nm $\lambda_2^{(n)}$ value in range $\pm$**15 nm from the real value for the sample** $\lambda_3^{(n)}$ value in range $\pm$**15 nm from the real value for the sample** $\lambda_4^{(n)}$ value in range $\pm$**15 nm from the real value for the sample** $\lambda_5^{(n)}$ value in range $\pm$**15 nm from the real value for the sample** $\lambda_6^{(n)}$ value in range 400-420 nm 0.2 pt for each (no more than 1.0)	0.1 0.1 0.2x5		

D2	m deriving method: $1/\lambda^{(n)}$ values analysis Searching for missing minima (using graph or $\Delta\left(1/\lambda^{(n)}\right)$ calculation or equivalent) m numerical values 0.2 for each correct value (no more than 1.2 in total) *Error* ± 1 *for m value*	0.4 0.4 0.2x6 0.1x6		
D3	D_Z value in range **1680 – 1920 nm**	0.3		
D4	m' values obtained from D2 Values for missing $\lambda_m^{(n)}$ obtained in range $\pm$ **15 nm from the real value for the sample**	0.1x2 0.4x2		
	Part D total	**4.5**		
E1	Scored only if t values in task C8 obtained Correct sample structure name selected (n-6) t values are in agreement with the table in Appendix (central minima are deep, peripheral minima are shallow)	0.2 1.0		
E2	Idea: structure selection using missing minima numbers **Points below are given only if the missing minima in task D4 are correct!** Correct sample structure name selected (hi5-5) Missing minima numbers are in agreement with the table in Appendix	0.3 0.2 0.8		
	Part E total	**2.5**		
	Total points	**20**		