理论题 1: Shockley-James 佯谬

1905 年，为了解决牛顿力学和麦克斯韦电磁理论之间的矛盾，阿尔伯特·爱因斯坦提出了狭义相对论。基于对狭义相对论的正确理解，使许多表面上的佯谬得到解决。那时候，讨论主要集中在电磁波的传播上。

在本题中，我们来解决一种不同类型的佯谬。W. Schockley 和 R. P. James 于 1967 年提出了一个相当简单的电荷系统，要理解这个系统中线动量的守恒性，需要作仔细的相对论分析。如果一个点电荷处在一个磁化强度变化的磁体附近，这个电荷会受到感应电力的作用，但这个磁体不会受到表观的反作用力。这个过程可以进行得相当慢，这样任何电磁辐射（以及任何由电磁波带走的动量）都可以忽略。这样的话，似乎我们得到了一个发射炮弹时没有反冲的大炮。

我们将对这个系统作分析，并要证明，在相对论力学中，一个复合体系可以具有非零的机械动量，而仍然保持静止。

第一部分：作用在点电荷上的冲量(3.3 分)

考虑一个半径为r，电流为I_1的圆形电流环，以及另一个更大的，半径$R \gg r$的圆形电流环，两个环共圆心，并且在同一平面上。

a. (1 分) 设电流环 2（较大的那个环）中的电流 I_2 在环 1 中产生的磁通量是 Φ_{B1} 。试给出比值$M_{21} = \Phi_{B1}/I_2$的表达式。M_{21}称为互感系数。

b. (0.8 分) 已知 $M_{12} = \Phi_{B2}/I_1 = M_{21}$ ，试求由于小环中的电流变化 $\dot{I}_1 = dI_1/dt$而引起的大环中的总感应电动势 ε_2 。忽略大环中的电流。*提示：感应电动势等于环中磁通量的变化率。*

c. (0.5 分) 你在(b)中得到的感应电动势是由感应电场的切向分量引起的。试求出在半径 R处切向电场 E的表达式（用电流变化率 $\dot{I}_1$表示）。

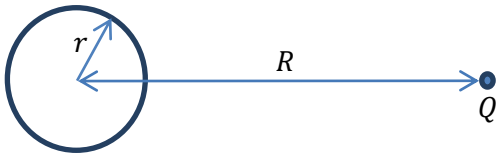


图 1: 圆形电流环与点电荷 Q。

现在我们拿走大电流环，代之以在半径 R 处放一个带电质点 Q ，如图 1 所示。可以假设，在所关心的时间段内该点电荷几乎不怎么动。

d. (1 分) 试求出在小电流环内电流从初值 $I_1 = I$ 变化到终值 $I_1 = 0$ 过程中，点电荷受到的总的切向冲量 Δp 。

第二部分：电流环受到的反冲(4.4 分)

现在，我们要用一个与前述形状不同的电流环，来理解电流环受到的反冲力的来源。

e. (1.1 分) 考虑一个空心管，其管壁为不带电绝缘材料，管长为 l，横截面积为 A，管内有电流 I。电流是由均匀分布于管内部的带电粒子运动所产生，每个粒子的静止质量为 m，带电荷为 q，其数密度为 n。假设所有带电粒子均以相同速度沿着管运动。考虑狭义相对论效应，试得出管内带电粒子的总动量 p。

f. (3.3 分) 考虑一个边长为 l 的正方形电流环。在与此环距离为 $R \gg l$ 处有一个点电荷 Q（见图 2）。此环中有电流强度为 I的电流流过 。在这里，与(e)中一样，我们采用的电流环模型是一个不带电的绝缘空心管子。载流子可以沿着环自由运动；在环的转角处，载流子与管壁发生弹性碰撞而作直角转向。忽略载流子之间所有的相互作用。并假设管子上同一横断面内所有载流子运动速度相同。假设环很重，其运动可忽略。计算环内载流子的总线动量 p_{hid}。这个动量称为“隐藏动量”。

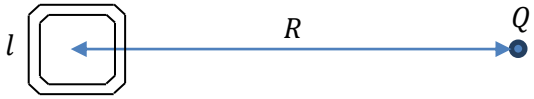


图 2: 正方形电流环与点电荷 Q.

当电流停止时，此线动量将转移到环上，环获得一个冲量，该冲量与点电荷 Q 受到的冲量大小相等，方向相反。这个冲量就是我们所要找的消失的反冲。（注意在初始状态下，电磁场中也有动量；这保证了整个系统的动量守恒。）

第三部份：总结有关结果(2.3 分)

g. (0.8 分) 电流环常用其磁矩 $\mu = IS$ 来表征，其中 I 为电流，S 为环面积。试用 μ， r， R 及 Q 来表示 (d) 部份的答案。同样，试用μ ，l， R及 Q来表示 (f) 部份的答案。注意有关电与磁常数的关系为：

$$\frac{4\pi k}{\mu_0} = \frac{1}{\varepsilon_0 \mu_0} = c^2$$

其中 c 是光速.

h. (1.5 分) 在更真实的模型中，电流环是一条导体电线，而点电荷 Q 所产生的电场不能穿越进入于导体内。假设电流仍然是由电线中的载流子传导的。试判断以下每个陈述句是否正确，在答题表中对判断结果画圈。**提醒**：你可以留下某些陈述句不作判断，但若其中一句判断错误， (h) 部分将不能得分。

A. (0.5 分) 电流环的线性动量是零。

B. (0.5 分) 随着电流环中的总电流由 I 变为零，载流子减速，在电线的导体材料中引起感应电流。因为产生了感应电流，点电荷Q 将不会获得净冲量。

C. (0.5 分) 随着电流从 I 变为零，由外部电荷在导线上感应出的表面电荷，将受到一个电场力。这样，电流环获得的冲量将与(f)中算出的一样。

理论题 2：嘎吱作响的门

嘎吱作响的现象很常见。开关门和橱柜时，粉笔在黑板上划过时，拉小提琴时，穿新鞋子在地上走时，汽车刹车时，还有日常生活中的很多其它场合都可以发现这种现象。在以色列，类似的现象每隔几十年就会导致一次剧烈的地震。这些地震来源于死海裂谷，而死海正好位于地壳上已知最深裂缝的正上方。

嘎吱作响现象的物理机制与弹性以及静摩擦系数和动摩擦系数之间的差异有关。在本题中，我们要探讨此机制，并具体地研究开门的例子。

第一部分：一般模型（7.5 分）

考虑如下系统（见图 1）：

一个质量为 m 的盒子与一个长的理想弹簧（劲度系数 k）相连，弹簧的另一端以恒定速度 u 被驱动。盒子和地面之间的静摩擦和动摩擦系数分别为μ_s和μ_k，且$\mu_k < \mu_s$。

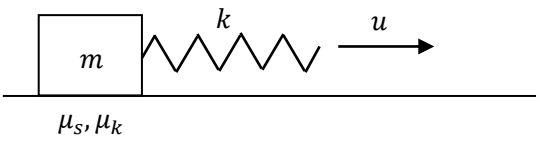


图 1：解释嘎吱作响现象的一般模型

我们要来探讨为什么这个系统可以有下面两种不同形式的运动：

1. 摩擦总是动摩擦。这被叫做**纯滑动**模式。

2. 动摩擦和静摩擦交替发生。这被叫做**粘滞-滑动**模式。

粘滞-滑动模式是常见的嘎吱声的来源。

a. （1 分）在初始时刻，$t = 0$，盒子在地面上以速度v_0滑动，弹簧的张力恰好与动摩擦力平衡。假设$0 < v_0 < u$. **弹簧伸长量**x将是时间t的振荡函数。

a1. (0.6 分)求此振子的振动周期T_0和振幅 A。

a2. (0.4 分)定性地画出$0 < t < 3T_0$时间间隔内，弹簧的伸长量与时间的关系$x(t)$。

b. （1.2 分）现在考虑$t = 0$时盒子静止，而弹簧的初始伸长量x与（a）中相同时的情况。定性地画出$0 < t < 3T$时间间隔内，盒子**相对于地面**的速度随时间的变化关系$v(t)$。其中，T是（新的）振动$x(t)$的周期。规定向右运动时速度v符号为正。在你的图上标出$v = u$的那条水平线。

c. （0.5 分）采用（b）的初始条件，求出充分长时间以后弹簧伸长量的时间平均值$\bar{x}$.

d. (2.4 分) 采用（b）的条件，求 $x(t)$的振动周期 T 。

一般来说，粘滞-滑动运动在驱动速度 u 很大时会停止。我们现在来讨论这个现象的可能机制之一。

e. (2.4 分) 假定在每一个振动周期中 T，一小部分能量会以热的形式在弹簧中耗散掉。令$\eta = |\Delta A/A|$为纯滑动模式下一个周期内能量耗散引起的相对振幅损失。对于$\eta \ll 1$的情况，当驱动速度超过某一临界速度u_c后，粘滞-滑动运动将不可能出现。请求出此临界驱动速度u_c。

第二部份：会嘎吱发响的门 (2.5 分)

门轴是一半径为 r，高度为 h，及厚度壁厚为 Δr 的中空金属圆筒。如图 2 所示，此金属圆筒的下端被放置于一个固定在墙上的金属台基上（两者的接触面是一个半径为 r 的圆环）。圆筒的底部与金属台基之间的静摩擦系数及动摩擦系数分别为 μ_s 及 μ_k ，且 $\mu_s < \mu_k$。圆筒的上部连到门上，而门可视作为刚体。一般的门都会用上二至三个这类的门轴，但门的重量只会集中于其中一个上，而就是这个门轴会发出嘎吱声。发出嘎吱声的门轴会通过金属圆筒的下底将质量为 M 的整扇门的重量加到金属台基上。

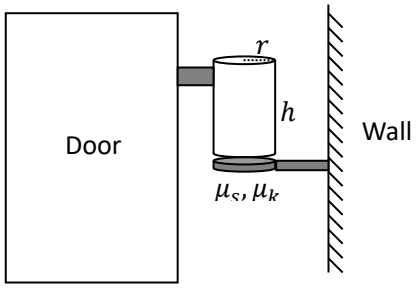


图 2： 门的示意图

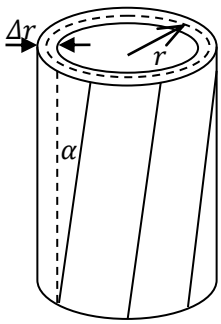


图 3：扭变后的门轴

构成门轴的金属圆筒并不是刚体 – 它可以在不改变其外形的情况下发生切向扭曲，其结果是（如图 3 所示），垂直线段会倾斜，与原来方向成大小为α的小倾角。由于形变而在圆筒底面面积元 dS 上产生的弹性力为：

$$dF = G\alpha dS,$$

其中，G 为剪切模量，为材料的一种力学性质。题中各量的数值取为：r=5mm, h=3cm, Δr =1mm, $\mu_s = 0.75$, $\mu_k = 0.55$, $G = 8 \times 10^{10} Pa$, M=30kg, $g = 9.8\, m/s^2$。你可以使用$\Delta r \ll r$的近似。

f. (1 分) 我们从平衡状态（零转矩）开始缓慢地转动门。对于很小的转动角度，请给出转矩系数$\kappa = \tau/\theta$的表达式，其中，τ是使门转过角度θ时需加在门上的转矩。

g. (1.5 分) 在角速度很小的情况下，当圆筒底部与台基间从相对静止转为滑动时，便会发出一个声脉冲。求：当此声脉冲的频率进入可听域$f = 20Hz$时，门转动的角速度Ω。假设圆筒做纯滑动振动时的频率$f_0 \gg f$。要求给出表达式和数值结果。

理论题 3：生日气球

图片 1 是一个在生日聚会上常见的长形橡皮气球。一个充气不完全的气球通常会分成两个具有不同半径的部分。在这个问题中，我们考虑一个简化的模型来理解这个现象。

气球具有均匀圆柱体的形状（除了末端之外），还有一个可以对气球充气的吹气口。所有过程都被当作室温下的等温过程来考虑。在所有时刻，气球内部的压强 P 都比大气压 P_0 只大一点，所以可以把空气当成不可压缩流体来对待。重力和气球的重量也可以忽略。充气过程是缓慢的、准静态的。在(a)-(e)部分，气球在它整个长度上被均匀充气。我们用 r_0 和 L_0 表示气球充气之前的半径和长度。

a. (1.1 分)气球一端由吹气口夹持，气球的其它部分自由悬挂着。确定纵向表面张力（沿平行于气球轴线的方向）和横向表面张力（沿气球圆形横截面的切向）之间的比值 sigma_L/sigma_t。

   橡皮膜的表面张力是，每单位长度的边界上，边界两侧的橡皮膜之间互相作用的力。

胡克定律是对小张力下真实弹性的线性近似。假设气球的长度仍然保持不变，为 L_0，而表面张力 sigma_t 线性依赖于充气率 r/r_0：

b. (1 分)在这些假设下，求出气球内部压强 P 对气球体积 V 的依赖关系的表达式。画出 P-P_0 随 V 变化的函数图像。从胡克弹性近似得到的最大充气压强 P_max 是多少？

实际中，由于充气率 r/r_0 很大（在图 1 中，观测到的典型值大约是 5），必须考虑橡皮的非线性行为，以及气球长度的变化。这些效应会使充气压强高于(b)部分预测的压强。在一个典型的气球中，sigma_t(r)的图像包括三个部分：

1. 对小的充气率，sigma_t(r)按胡克定律的方式增长。
2. 在 r-r_0~r_0 附近，气球的长度 L 开始增加，而且 sigma_t(r)达到一个长的平台区，在这个区域内增长十分缓慢。
3. 在充气率很大的时候，橡皮开始强烈抵抗任何进一步的伸长，这会导致 sigma_t(r)急剧增长。

图 2 描绘了这样的行为。

c. (1.3 分)对行为遵从图 2 的均匀充气气球，定性地画出压力差 P-P_0 随 V 变化的函数图像。在你的图中标出所有局部极值点。也要标出对应于 r=1cm 和 r=2.5cm 的点。找出在这两点处 P-P_0 的值，要求精确度为 10%。

为了探索你在(c)部分中所发现的行为的结果，对一个均匀充气的气球，我们用一个三次函数来对 P(V)作近似：

其中 a, b, c 和 u 是正的常数。假设体积 u 大于未充气时气球体积 V_0；同时假设 c 足够大，使得函数(2)在整个 V>V_0 的物理范围内都是正的。

d. (0.6 分) 计算将气球从体积V_i膨胀到V_f连续充气过程中所作的功 W。用 a, b, c 和 u 来表示你的答案。

气球被连接在一个巨大的空气源上，气源的气压 P 可控。事实上，气压 P 的值可能对应多个不同的体积值 V。如果在保持气压恒定的情况下对气球进行微小的扰动（比如由外力进行局域拉扯），气球有可能突然改变其体积，进入另一个态。状态转变后整个系统末状态能量应该更低（系统包括气球、大气和保持气球气压 P 的装置）。如果气压从P_0开始缓慢增加，并始终伴随着足够的微扰，这种突然的体积改变便会在气压达到某一值P_c时出现。这种不连续现象在自然界中十分普遍，有时被称作“相变”。

e. (1.7 分) 找出P_c，相变前的体积V_1，相变后的体积V_2。用 a, b, c 和 u 来表示你的答案。

考虑更现实的情况，由一个小孩来对气球进行充气，该情况下将不能提供足够的空气使得气球的体积突然改变。空气将逐渐进入气球，这等效于控制气球的体积而不是其中的气压。在这种情况下，气球将表现出另一种新的行为。如果有利于整个系统进入最低能量状态，气球将分裂为两个半径不同的部分（在对其有足够

扰动的情况下），两部分的长度会逐渐变化。分裂界面上需要聚集能量，我们在此忽略此能量。我们在此也忽略界面的长度（这些近似对于一个很长的气球来说是合理的）。

f. (1 分) 定性绘出气压差 $P - P_0$ 对体积 V 的关系线图，气球的分裂亦须列入考虑。在线图中指出气压 $P_c - P_0$ 及体积 V_1 与 V_2。

g. (1.4 分) 气球于某体积范围下能分裂为两个部分。找出较细部分的长度 L_{thin} 总空气体积 V 的函数 $L_{thin}(V)$ 。用体积 V_1 、 V_2 及较细部分的半径 r_1 表示你的答案。

h. (1.2 分) 气球于某体积范围下能分裂为两个部分。求使较细部分转变为较粗部分所需的单位长度潜热功 $\Delta W / \Delta L_{thin}$ 。

i. (0.7 分) 在现实充气过程中，气球体积逐渐增加，且外来扰动很小。在此情况下，定性绘出另一气压差 $P - P_0$ 对体积 V 的关系线图。假设在充气过程中的某个阶段，气球仍会分成两部分。在标示体积 V_1 于水平轴上标示出该阶段。

实验题 1
导体在振荡磁场中的悬浮

在强度足够大的振荡磁场的作用下，金属导体是可以悬浮起来的。悬浮现象是由于作用在导体中的电流的平均磁力不为零而产生的。电流本身也是由振荡磁场感应而产生的。磁场和感生电流之间的相位差导致了洛仑兹力的平均值大小不为零。这个相位差是因导体中电流环路的自感而产生的。

在本实验中，我们将研究上述现象，并通过测量由通有振荡电流的螺旋管施加在铝环上的竖直方向的力，来导出铝环的自感系数。为了方便测量，作用在环上的平均力方向将会指向下方，故不会观察到悬浮现象。

在你的桌子上，应该放着以下物品 （图 1）：

（1） 一个工作频率为 50Hz 的交流电源。该电源有两对输出端。其中两个较小插座孔（图 1a）输出电压约为 24V；这两个插孔只能用于对螺线管供电。另外两个较大的插座孔（图 1b）输出电压约为 0.7V。只有按住绿色按钮（图 1c）时，电源才会有输出。这是为了防止意外地使电源一直开着而导致电源过热（下文会有解释）。电源开启时，红色指示灯会亮。

（2） 一个圆柱形带有铁芯的螺线管。该螺线管应当接在电源的 24V 输出端上。螺线管的位置可调，通过螺丝杆可以上下调节其位置。螺丝杆的竖直螺距为$h = 1.41mm$。

（3-5）三个由完全相同的材料（一种铝合金）制成的金属环。其中，一个环是闭合的。另一个金属环尺寸与闭合环一样但切掉了一小段，成为一个开口环。最后一个金属环也是开口环，而且比前两个环细很多。

（6-7）两个万用表。分别被作为电压表和电流表来使用。交流电压表的灵敏度为 0.1mV。电流表的量程为 20A（交流模式下也是）。**注意：**在交流模式下，万用表将显示被测量的方均根值，即，其振幅除以$\sqrt{2}$。详细用法见图 3 及图 4。

（8） 使用电池供电的数字天平，灵敏度为 0.01g。当天平受到快速振荡变化的力作用时，它将显示出力的时间平均值。**注意：**天平上有“tare”（去皮）这一按钮。在天平负重时，按下这一按钮，天平示数将被调校归零。见图 2。

（9） 一个 8cm×7cm×7cm 的聚苯乙烯泡沫块。该泡沫用于安放金属圈。

（10） 带有各种不同接头的电线。

（11）一把尺子。

（12） 毫米绘图纸。

（13） 一盏照明用的台灯。

警告： 当置于螺线管的磁场中时，闭合金属圈上所带电流的强度将会很大。金属圈将被此电流加热。温度变化将使金属圈电学性质有微小改变。为了避免该误差，不要对螺线管长时间通电。

特拉维夫的重力加速度大小为$g = 9.80 \pm 0.01\mathrm{N/kg}$

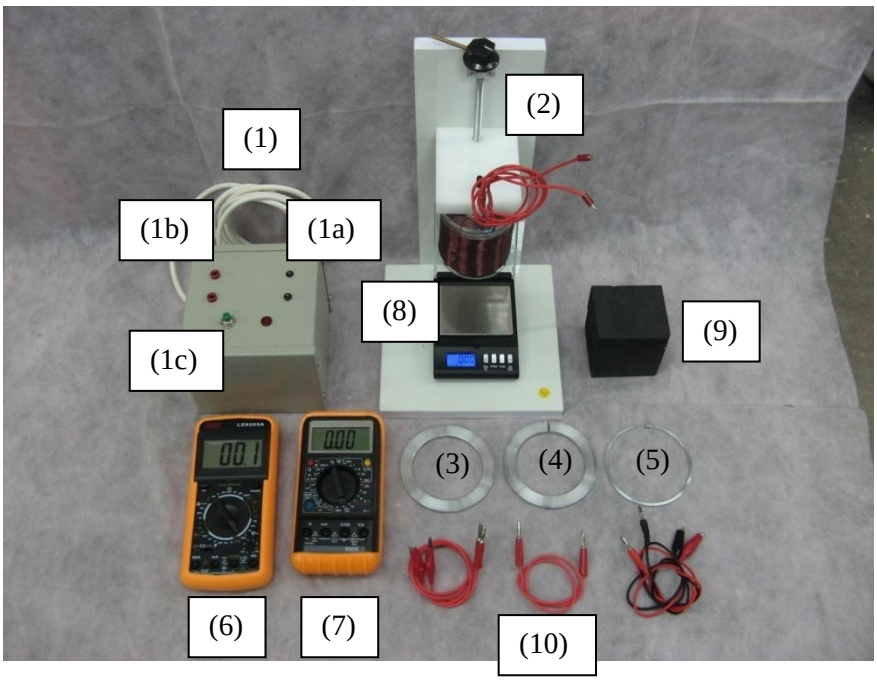


图 1：实验装置

(1) 50Hz 电源。
（1a）24V 端口。
（1b）0.7V 端口。
（1c）电源按钮。
(2) 安装在竖直螺杆上的螺线管。
(3) 宽的闭合环。
(4) 宽的开口环。
(5) 细的开口环。
(6) 电压表。
(7) 电流表。
(8) 数字天平。
(9) 聚苯乙烯泡沫块。
(10) 导线。



图 2：数字天平
（1）On/Off 开关键。
（2）“Tare”去皮键—可将当前的重量设为 0.

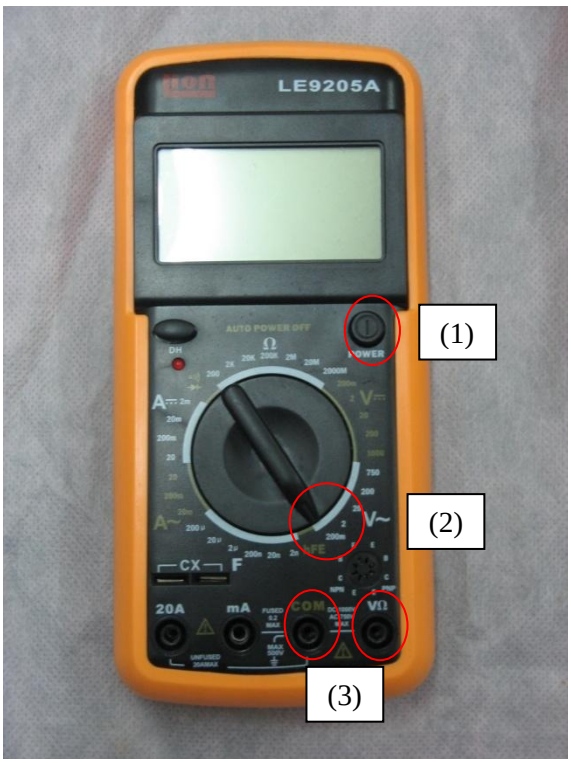


图 3：电压表

(1) On/Off 开关键
(2) 拨动开关置为 200mV 交流。
(3) 把你的导线接到“COM”和”V/Ω”端口。

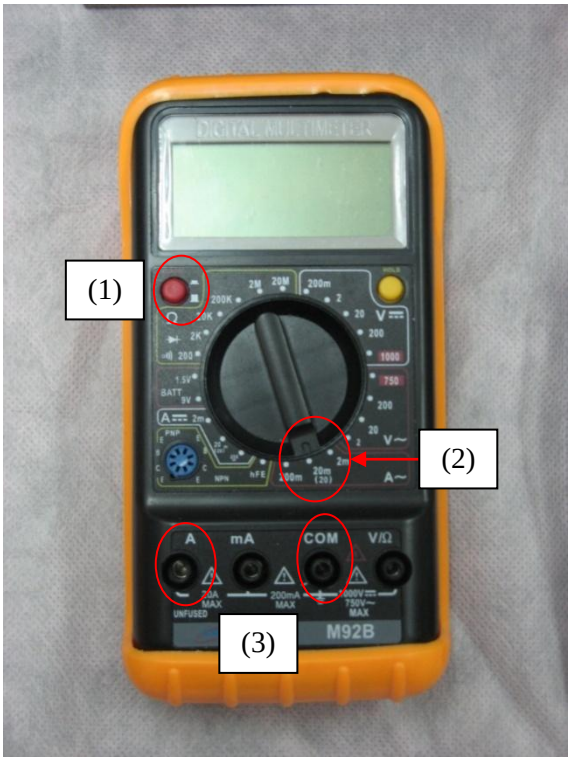


图 4：电流表

（1） On/Off 开关键
（2）拨动开关置为 20mA/20A 交流。
（3）把你的导线接到“COM”和”A”端口。

理论（1.3 分）

考虑放置于螺线管磁场内的半径为 r 的导体环。螺线管的对称轴与导体环的对称轴重合。导体环的电感记作 L，电阻记作 R，螺线管电流的角频率记作ω。z 轴方向定义为沿着螺线管与导体环的对称轴的方向。

仅仅在题目的这一部分，你才可以忽略掉导体环的磁场对螺线管及其铁芯的微小作用。同时，也可以忽略导体环的厚度。

在此部份，我们将用到法拉第定律，及磁场的高斯定理：

- 法拉第定律：变化的磁通量在回路上产生的感生电动势是$\epsilon = -d\Phi_{\mathrm{B}}/dt$.
- 磁场的高斯定理：穿出闭合面的总磁通量为零。

置于柱对称的磁场$\vec{B}$中的电流环所感受到的合力为：

$$F(t) = -2\pi r I(t) B_r(t)$$

其中，I为环中的电流，B_r为环附近外磁场的径向分量（沿环半径方向）。力 F 的正方向指向下方——沿z方向。电流 I 的正方向如图 5 所示。

a．（0.2 分）考虑有外部振荡磁通量$\Phi_{\mathrm{B}}(t) = \sqrt{2}\Phi_{\mathrm{B}}^{\mathrm{rms}} \sin(\omega t)$穿过环的情形．求：$\epsilon(t)$——仅仅由外部振荡磁通量所产生于圆环上的感生电动势；$I(t)$——环内的感生电流，将$I(t)$用$\Phi_{\mathrm{B}}^{\mathrm{rms}}, L, R, \omega$和$t$表示.

提示：对于既有电感又有电阻的交流电路元件，其上电动势的振幅ϵ_0以及电流振幅I_0的关系为：$\epsilon_0 = I_0\sqrt{\omega^2 L^2 + R^2}$, 并且，电流相对于电动势延迟的相位为$\delta = \tan^{-1}\frac{\omega L}{R}$.

b．（0.6 分）求B_r，（以r、$\frac{\mathrm{d}}{\mathrm{dz}}\Phi_{\mathrm{B}}$表示，$z$ 轴垂直于圆环所在平面）。

c．（0.5 分）证明：$\langle F\rangle = \alpha\frac{L}{(R^2+\omega^2L^2)}\cdot\frac{d(\epsilon_{\mathrm{rms}})^2}{dz}$. 这里，$\langle F\rangle$为$F$的时间平均值，$\epsilon_{rms}(z)$为当环处于高度$z$时，环上感生电动势的方均根值 (即振幅除以$\sqrt{2}$)。求出：常量α.（如果你没有求出 α , 在以后的各部分中，请把 α 取为 1)。

提示：以下公式可能有用：

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\langle(\sin(\omega t))^2\rangle = \langle(\cos(\omega t))^2\rangle = \frac{1}{2}$$

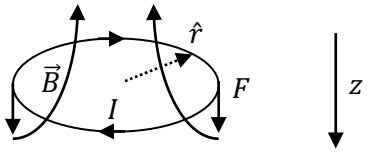


图 5：金属环在螺线管的磁场中。

测量（5.1 分）

在下面所有测量和分析中，给出的结果精度越高，所得的分数也将越高。在你的所有测量和结果中，给出误差估计。

电阻测量（2.6 分）

在这部分，你必须使用交流电源的 0.7V 端口。若用提供的导线把两个 0.7V 的端口短路，得到的电流会在 5A 至 10A 之间，依赖于接触的情况。提醒：相比较于那 2 条较长的导线而言，那 3 条较短导线与电流表接触得更好。**警告**：为了避免装置过热，此时不要使用 24V 的端口。

d. （1.3 分）测出细环的电阻R_{thin}，在答题表中画出你的接线图。

*提示：这里的每一个环的电阻都比*0.1Ω *小得多。而且，对这个细环而言，相对于其电阻值，你可以忽略感抗。*

e. （1.3 分）确定出闭合环的电阻 R。在必要时需添加一些测量。

感应电动势的测量（1.5 分）

f. （1.5 分）把螺线管与电源的 24V 端口相连。安放好宽的开口环，使环的轴线与螺线管的轴线重合。在不同的高度 z （即离螺线管不同距离处），测量环上的感应电动势ϵ_{rms}. 在答题表给定的表格中记录你的测量值。画出ϵ_{rms}随 z 变化的函数图（包括趋势线）。

力的测量（1 分）

g. （1 分）安放好宽的闭合环，使环的轴线与螺线管的轴线重合。在不同的高度 z （即离螺线管不同距离处），测量环受到的磁力的时间平均值$\langle F\rangle$. 在答题表给定的表格中记录你的测量值。

分析（3.6 分）

h. （1.4 分）在问题“g”中，你测量了不同 z 值的作用力，在这些 z 值处，求出ϵ_{rms}^2对z的微商的绝对值$|d\epsilon_{rms}^2/dz|$. 在答题表给定的表格中记录你的值。这一部分不需要误差估计。

i. （2.2 分）使用线性图分析你的结果，来求出闭合环的自感系数 L。你可以利用$\omega L < R$这一事实。

提醒：尽管闭合环有明显的厚度，你在问题“c”中导出的公式仍然有较高的精度。可以把上述结果用作宽环电感的操作型定义。

提示：当环太靠近铁芯时，测量结果将发生偏差。试在你的分析中避开这个麻烦。

实验试题 2
光学“黑盒子”

电视和计算机显示屏技术近年来飞速发展。今天的显示屏多为一个彩色液晶屏（LCD），外加一个白色的背光源。在这个实验中，我们将研究一种有希望用于背光照明型 LCD 的塑料材料。

实验

在你的桌子上，应有以下物品（见图 1）：

（1） 实验样品——一块放在幻灯片框中的塑料材料。此**样品很脆弱，请务必不要触碰。如要调节样品的位置，请握住其台座来移动。**

（2） 幻灯片框被安放在一个台座上。此台座上有一个手柄，通过手柄可以精细旋转上面的样品。**请勿将幻灯片框移出台座。**

（3） 一个使用白光发光二极管（LED）的电筒。电筒的开关按钮在其非出光端部。**请勿将电筒与激光笔器混淆（见图 2）。**

（4） 一支发出红色激光的激光笔。**该激光笔上贴有一警告标签。请勿将激光笔与白光电筒混淆（见图 2）。**通过前后推移出光端的黑色笔帽可以将激光笔开启/关闭。**请不要尝试拆卸笔帽——这样做比较危险，并可能损坏激光笔。**激光笔使用电池作电源，连续使用一小时后激光笔强度便会开始下降。故请非必要时不要开启激光笔。激光笔的波长是$\lambda = 652\text{nm} \pm 2\text{nm}$。

（5） 一个可用于分别固定两种光源的支座。**实验开始时，固定在支座上的是电筒，放置在桌上的是激光笔。**

（6） 有木质支座的白色屏幕，屏上覆有毫米绘图纸。在屏幕中央有一穿孔。你可以在屏幕上作记号。

（7） 一个可在木质滑轨上前后滑动的木质标志杆。你可以在木质滑轨上作记号。

（8） 一把卷尺。

（9） 一把直尺。

（10）毫米绘图纸。

（11）一盏供照明使用的台灯，你可根据需用打开或关闭。

激光笔安全须知：

1. **请勿直视激光束！**
2. **要小心从金属表面上反射回来的激光束。**
3. **不要将激光笔指向他人。**
4. **不要尝试修理或拆卸激光笔。如需帮助，请呼叫监考老师。**

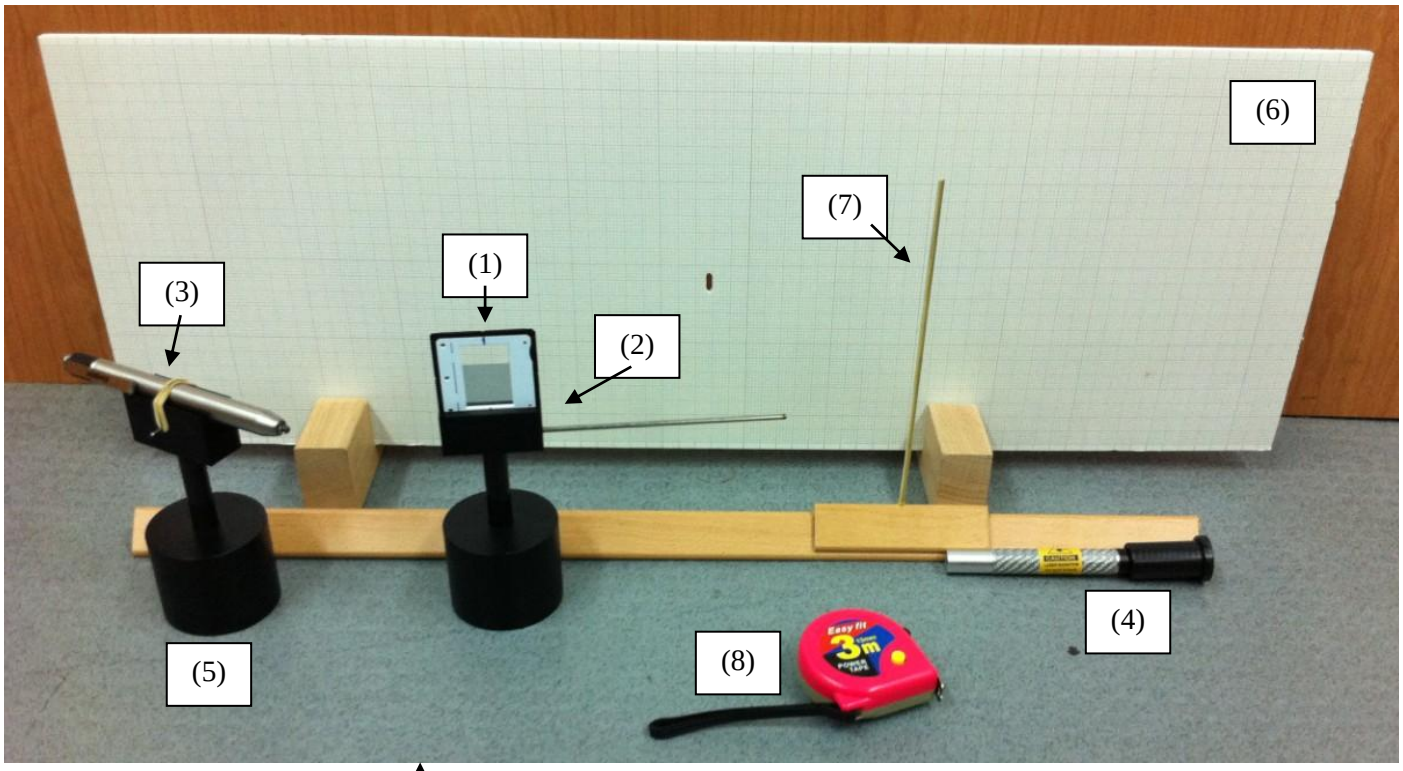


图 1：实验装置清单

（1）放置在幻灯片框内的塑料样品。
（2）样品台座。
（3）LED 白光手电筒。
（4）红色激光笔。
（5）光源台座。
（6）覆有毫米刻度绘图纸的光屏。
（7）放置在导轨上的木制标志杆。
（8）卷尺。

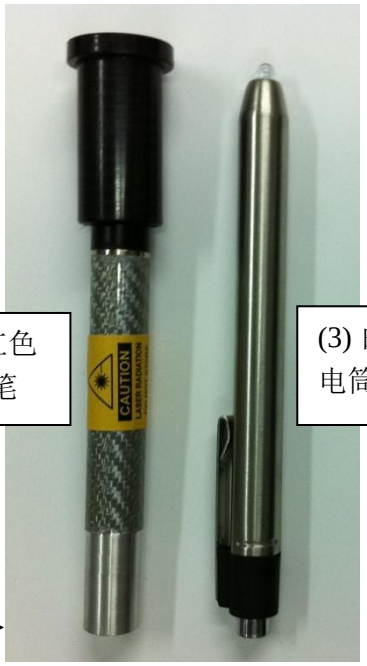


图 2：两个光源的特写。

第一部分 — 理论（0.4 分）

a. （0.4 分）一支光线被夹角为φ的两面平面镜反射（图 3）。请找出入射和出射光线的夹角γ. 假设所有光线均在一个与两平面镜相交线垂直的平面内。

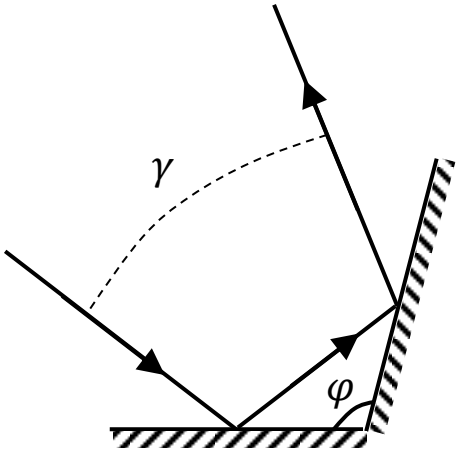


图 3: 一束光线被两个镜面反射。

第二部分 — 使用白光光源作测量（6.1 分）

你要用白光手电筒做光源来观察样品的透射和反射特性。图 4 给出了透射和反射两种观察方式的装置放置参考示意图。注意：照明光从样品的不同面入射时，你会看到不同的结果。

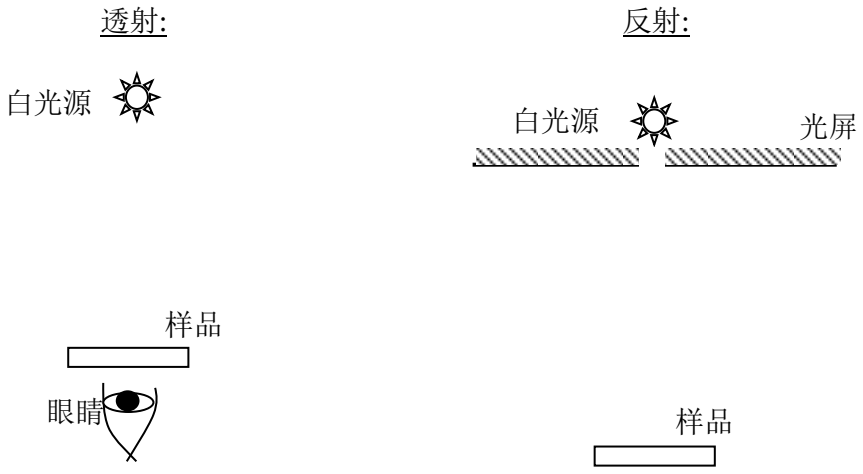


图 4: 用白光时的设备放置参考示意图。

警告：为了观察透射光，你将不得不通过样品直视手电筒的光线。**对激光，你千万不要这样直视！**另外，不要长时间直视手电筒的光线。

b. （0.5 分）图 5 是样品的 4 种可能微观结构的示意图。n 表示该塑料样品的折射率。从中选出最符合你的观察结果的微观结构。注意：图中所示的 5 个周期只是用来说明。实际情况是：d 很小，样品包含很多个周期。

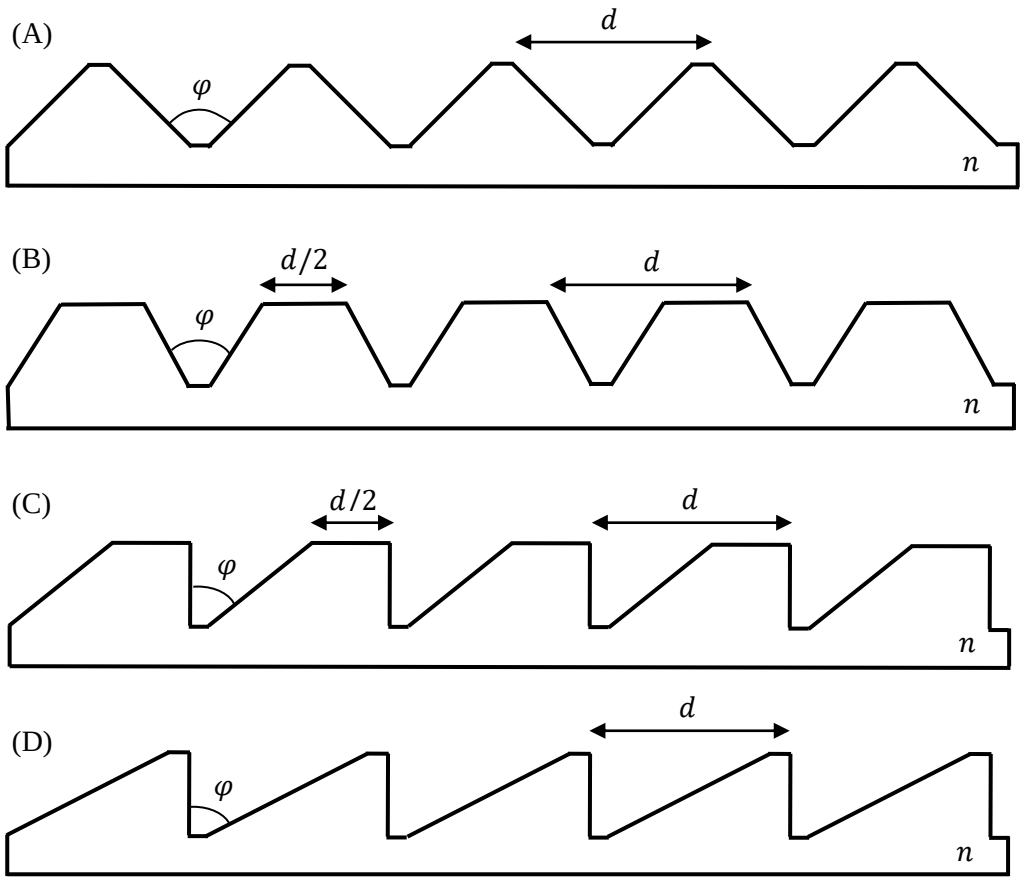


图 5: 样品的几种可能微观结构。

c. （0.8 分）给出样品的 φ 角，并估计其误差。

d. （0.5 分）当一束白光从样品的某一面垂直入射时，在透射光中，在光源右方靠近光源的位置，可观察到如下图所示的暗淡图样（图 6）。图中，“R”、“G”和“B”分别表示红、绿和蓝。注意：对此图样的观察可能比较困难，因此不要求对它进行测量。

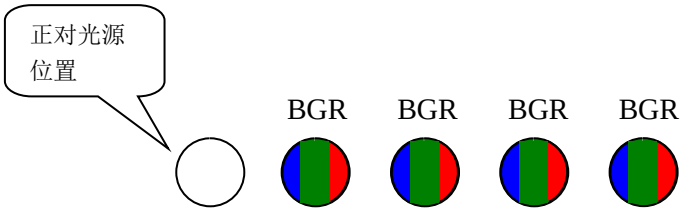


图 6: 光源附近的暗淡图样。

视线再往右移，你会观察到一个明亮得多的图样（图 7）。

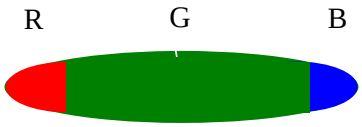


图 7: 光源的右方更远处的主透光图样。

选出正确的选项：

A. 所有彩色图样都源于干涉。

B. 所有彩色图样都是由于 n 依赖于波长而产生。

C. 图 6 中的彩色图样都来自干涉，而图 7 的图样则由于 n 依赖于波长而产生。

D. 图 6 中的图样由于 n 依赖于波长而产生，但图 7 的彩色图样来自干涉。

e. （1.4 分）如问题“d”中一样放置样品和光源，测量图 7 所示意的主透光图样中紫光（在光谱的最远蓝端）的偏向角δ_0. 偏向角在图 8 中定义。记录下所有中间测量结果，给出误差估计。

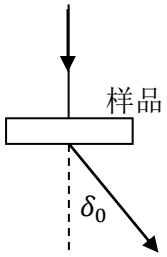


图 8: 偏向角 δ_0.

f. （1.4 分）不同的入射角照射样品，主透光会有不同的偏向角。测量主透光图样中紫光的最小偏向角δ_{min}（仅有一个最小偏向角）。记录下所有中间测量结果，给出误差估计。

g. （0.8 分）利用问题“c”中得到的角度φ，用δ_0 或 δ_{min} 把样品的折射率 n 表示出来。你可以用到光线传播的可逆性和只存在一个最小的偏转角 δ_{min} 这个事实。

h. （0.7 分）对紫光，给出样品的折射率n_v 并作误差估计。

第三部分 — 用激光笔作测量（3.5 分）

拿掉光源台上的电筒，并以激光笔代替。你可用白色屏幕来观察透射及反射的图案，如图 9 所显示。激光笔电量有限——除非必须，否则不要维持开启状态。在调光路时，转动激光笔方向会更容易。

警告：切勿直视激光束及其反射光束！切勿透过样品来观察激光——请以提供的光屏来观察。



图 9: 对于激光，设备放置参考图。



当你轻微旋转样品，可以在屏上观察到亮纹的亮度会强弱交替变化。某些亮纹的变暗，是由样品中同一“锯齿”上的不同区域之间的相消干涉引起的。

i. （1 分）选用图 9 中的一款观察装置，用激光笔垂直照向样品。记录所观察到条纹的偏向角θ作为条纹号 m 的函数。定义图案中心为$m = 0$。将有关记录填写于答题表中所提供的表格。记录所有中间测量结果。给出误差估计。

j. (1.5 分)利用一线性图，求样本中两个相邻锯齿的间距 d. 此图不要求标出误差棒。给出 d 的误差估计。

k. (1 分) 利用你在问题“g”所导出的公式，求出样品在激光红光波长下的折射率 n_r。记录新添加的所有测量数据。给出误差估计。**警告：切勿透过样品观察！请用提供的光屏来观察。**

Theoretical Question 1: The Shockley-James Paradox

SOLUTION

a. The magnetic field created by the large loop at its center is:

$$B = \frac{\mu_0 I_2}{2R}$$

Since $r \ll R$, this is the field throughout the area of the small loop. Therefore, the flux through the small loop is given by:

$$\Phi_{B1} = \pi r^2 B = \frac{\pi \mu_0 r^2 I_2}{2R}$$

The mutual inductance is then given by:

$$M_{21} = \frac{\pi \mu_0 r^2}{2R}$$

b. Since $M_{12} = M_{21} = M$, we have:

$$\Phi_{B2} = M I_1 = \frac{\pi \mu_0 r^2 I_1}{2R}$$

Taking the derivative with respect to time, this becomes:

$$\varepsilon_2 = \frac{\pi \mu_0 r^2 \dot{I}_1}{2R}$$

c. The EMF is work per unit charge, while the electric field is force per unit charge. Therefore:

$$E = \frac{\varepsilon_2}{2\pi R} = \frac{\mu_0 r^2 \dot{I}_1}{4R^2}$$

d. The electric field from part (c) leads to a force:

$$F = EQ = \frac{\mu_0 r^2 Q \dot{I}}{4R^2}$$

Integrating over dt (and disregarding the sign), we get the impulse:

$$\Delta p = \frac{\mu_0 r^2 I Q}{4R^2}$$

e. The current can be written as:

$$I = nAqv$$

where v is the charge carriers' velocity. We therefore have:

$$v = \frac{I}{nAq}$$

The momentum is then given by:

$$p = \gamma mnAlv = \frac{mnAlv}{\sqrt{1 - v^2/c^2}} = \frac{mIl}{q}\left(1 - \left(\frac{I}{nAqc}\right)^2\right)^{-1/2}$$

where γ is the Lorentz factor associated with v.

f. The hidden momentum is due to the charge carriers in the two vertical sides of the loop. Let m be the mass of the charge carriers, let q be their charge, and let $\Delta U = kQql/R^2$ be the potential energy difference for a charge carrier between the two sides. Denote the longitudinal densities and velocities of the charges in the two sides by λ_1, v_1, λ_2 and v_2. Let γ_1 and γ_2 be the appropriate Lorentz factors. From the constant value of the current, we have:

$$q\lambda_1 v_1 = q\lambda_2 v_2 = I$$

Energy conservation for the charge carriers passing from one side to the other reads:

$$(\gamma_2 - \gamma_1) \cdot mc^2 = \Delta U$$

The total momentum now reads:

$$p_{hid} = p_2 - p_1 = ml(\gamma_2\lambda_2 v_2 - \gamma_1\lambda_1 v_1) = \frac{mIl}{q}(\gamma_2 - \gamma_1) = \frac{Il\Delta U}{qc^2} = \frac{kQIl^2}{R^2c^2}$$

Note that all the microscopic quantities m, q, λ_i and v_i have dropped out.

g) In part (d), the magnetic moment is $\mu = \pi r^2 I$, and we get:

$$\Delta p = \frac{\mu_0 Q\mu}{4\pi R^2}$$

In part (f), the magnetic moment is $\mu = l^2 I$, and we get:

$$p_{hid} = \frac{kQ\mu}{R^2c^2} = \frac{\mu_0 Q\mu}{4\pi R^2}$$

We see that the results are identical.

h) The answer is (A)+(C). (A) is true because ΔU between the near side and the far side of the loop vanishes. (B) cannot be true, because the back-reaction of the induced charges on the external charge is a higher-order effect; for instance, it involves higher powers of Q. Then the conservation of center-of-mass velocity requires that (C) is true.

Theoretical Question 2: Creaking Door

SOLUTION

a1. The motion here is pure sliding under a constant kinetic friction. This is harmonic motion with a displaced equilibrium point. The angular frequency is given by:

$$\omega_0 = \sqrt{k/m}$$

From here, the period is:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

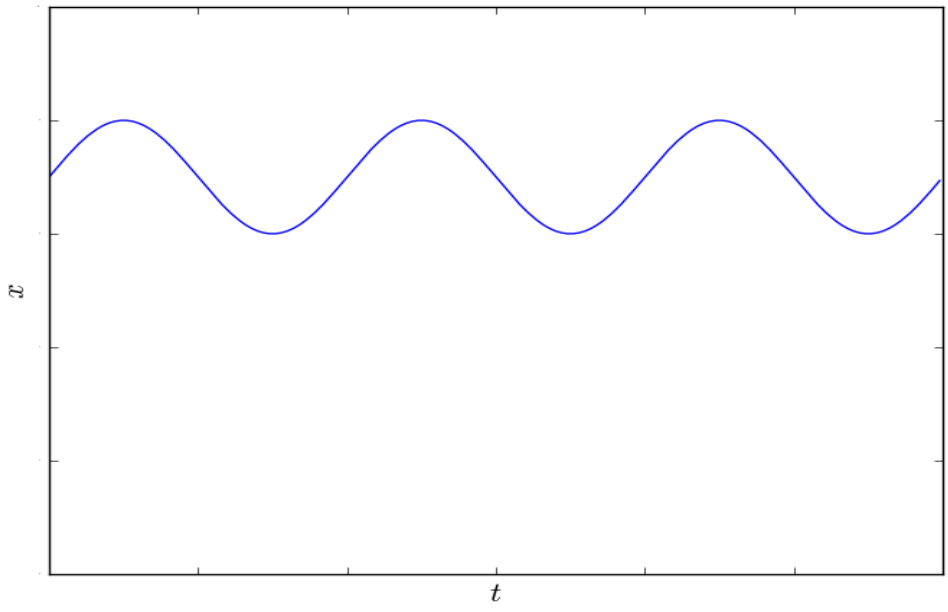
The initial slope is given by:

$$\left(\frac{dx}{dt}\right)_0 = u - v_0$$

Therefore, the amplitude of oscillations is:

$$A = \frac{(dx/dt)_0}{\omega_0} = (u - v_0)\sqrt{\frac{m}{k}}$$

a2. The graph is sinusoidal, as shown below.

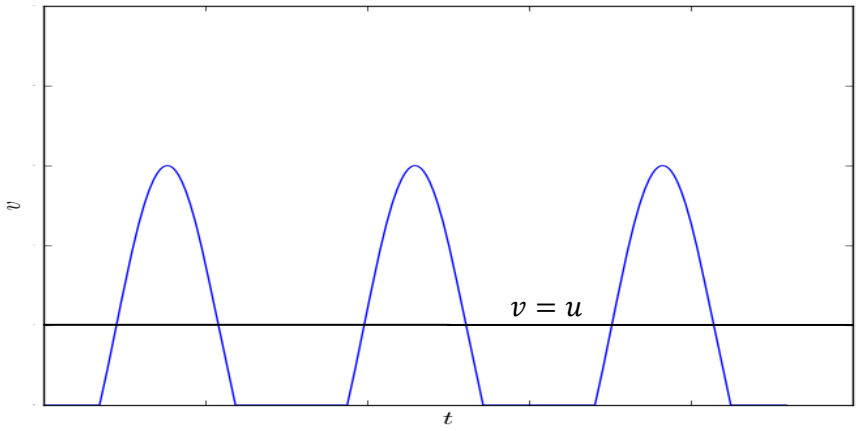


The initial point is given by:

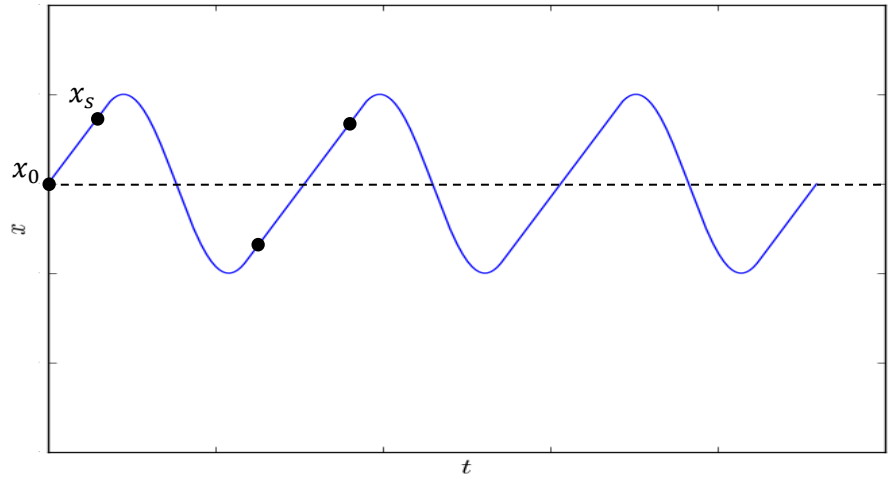
$$x_0 = \frac{\mu_k m g}{k}$$

This is the equilibrium point of the sine function. The students are not required to find this equilibrium point, but they are required to understand that it is positive.

b. This will be a stick-slip graph. The ”humps” are sinusoidal, with a non-continuous derivative at their intersections with the horizontal segments. The peaks of the humps are higher than $v = u$, since u must be the *average* velocity of the box. In fact, they are also higher than $v = 2u$, but this is not required from the students.



c. Let’s pass into the reference frame of the driven end of the spring. The position of the box is then given by minus the elongation x. The motion is an oscillation around the equilibrium position x_0. The slip phase is sinusoidal as in part (a), while the stick phase corresponds to motion with a constant velocity $-u$. The stick phase ends when the elastic force balances the static friction, i.e. at $x_s = \mu_s m g/k$, and starts again at the symmetric point with respect to x_0.



We see that the average elongation is again the sine’s equilibrium point:

$$\bar{x} = x_0 = \frac{\mu_k mg}{k}$$

d. Again, let us pass into the reference frame of the driven end of the spring. During the stick phase, the box traverses a distance of:

$$2(x_s - x_0) = 2(\mu_s - \mu_k)mg/k.$$

Its velocity during this phase is u, so the duration of the stick phase is:

$$t_{stick} = \frac{2(\mu_s - \mu_k)mg}{ku}$$

The slip phase is a sinusoidal motion around x_0 with angular frequency ω_0. The sinusoidal period is missing a phase of 2φ, where φ is given by the ratio of initial position and initial velocity with respect to the equilibrium point:

$$\tan\varphi = \frac{\omega_0(x_s - x_0)}{u} = \frac{(\mu_s - \mu_k)g}{u}\sqrt{\frac{m}{k}}$$

Then the length of the slip phase is:

$$t_{slip} = T_0\left(1 - \frac{\varphi}{\pi}\right) = 2\sqrt{\frac{m}{k}}\left(\pi - \tan^{-1}\left(\frac{(\mu_s - \mu_k)g}{u}\sqrt{\frac{m}{k}}\right)\right)$$

And the total period is:

$$T = t_{stick} + t_{slip} = 2\sqrt{\frac{m}{k}}\left(\frac{(\mu_s - \mu_k)g}{u}\sqrt{\frac{m}{k}} + \pi - \tan^{-1}\left(\frac{(\mu_s - \mu_k)g}{u}\sqrt{\frac{m}{k}}\right)\right)$$

e. Consider again stick-slip motion in the reference frame of the driven end of the spring. During the sinusoidal slip phase, the sine's amplitude will decrease due to the dissipation. At the beginning of the slip phase, the velocity is $-u$, while the sine is at the phase φ, which we found in the solution to the previous part. Thus, the sine's velocity amplitude is $u/\cos\varphi$. For periodic stock-slip to occur, the sine must return to the slope $-u$. Due to the dissipation, this will happen at a phase larger than $2\pi - \varphi$. In other words, dissipation shortens the stick phase. The critical case is when stick phase shortens to zero. This will happen if the sine reaches the slope $-u$ precisely at the equilibrium point, i.e. at the phase 2π. If the slope at 2π is less steep than $-u$, the box will continue its damped sinusoidal motion without ever reaching a stick phase again.

If it is to be killed by weak dissipation, the stick phase must be short to begin with. This corresponds to a large u. The slip phase then takes up almost an entire period of the sine wave. Thus, to a good approximation, the amplitude loss during the slip phase is given by η. The critical point is when the velocity amplitude drops from $u/\cos\varphi$ to u during one period:

$$\eta = \left|\frac{\Delta A}{A}\right| = \left|\frac{u/\cos\varphi - u}{u/\cos\varphi}\right| = 1 - \cos\varphi \approx \frac{\varphi^2}{2}$$

where the LHS is the change in the amplitude due to dissipation over one period. Using the results from (d) in the limit of small φ, we get:

$$\eta = \frac{m(\mu_s - \mu_k)^2 g^2}{2ku_c^2}$$

$$u_c = (\mu_s - \mu_k)g\sqrt{\frac{m}{2k\eta}}$$

Another derivation method based on the same reasoning is to use explicitly the initial amplitude A of the harmonic motion:

$$u_c = \omega_0 A(1-\eta), \qquad A^2 = (x_s - x_0)^2 + (m/k)u_c^2$$

A third method is to consider energy losses $|\Delta E/E| = 2\eta$ in the reference frame of the spring's driven end:

$$2\eta \cdot \frac{1}{2}mu_c^2 = \frac{1}{2}k(x_s - x_0)^2$$

f. For small rotations, the lower edge of the cylinder will remain stuck to the base. When the cylinder is deformed by an angle α, a point on its upper edge shifts by a distance $h\alpha$. This corresponds to a rotation angle $\theta = h\alpha/r$ of the door around the cylinder's axis. The shear force on an area element dA of the base is given by:

$$dF = G\alpha dA = \frac{Gr}{h}\,\theta dA$$

The corresponding torque is:

$$d\tau = rdF = \frac{Gr^2}{h}\,\theta dA$$

Summing over the contact area with the base, the total torque is:

$$\tau = \frac{Gr^2}{h}\,\theta \cdot 2\pi r\Delta r = \frac{2\pi G r^3 \Delta r}{h}\,\theta$$

Therefore, the torsion coefficient is:

$$\kappa = \frac{2\pi G r^3 \Delta r}{h} \approx 2000\text{Nm}$$

The numerical result is not required from the student. Any expression which reduces to the one above in the limit $\Delta r \ll r$ will be accepted.

g. We neglect the duration of the slip phase. Using the results of section (d) with M instead of m and rotation instead of linear motion, we get:

$$t_{stick} = \frac{2(\mu_s - \mu_k)Mgr}{\kappa\Omega}$$

$$\Omega = \frac{2(\mu_s - \mu_k)Mgr}{\kappa t_{stick}} = \frac{2(\mu_s - \mu_k)Mgrf}{\kappa} = \frac{2(\mu_s - \mu_k)Mghf}{\pi G r^2 \Delta r} = 1.12 \cdot 10^{-2}\text{s}^{-1}$$

Any expression which reduces to the one above in the limit $\Delta r \ll r$ will be accepted. Numerical results from such different expressions may vary significantly, since $\Delta r/r = 0.2$ is not really negligible. Each numerical result should be checked against its expression.

Experimental Question 1: Levitation of Conductors in an Oscillating Magnetic Field

SOLUTION

a. Using Faraday's law:

$$\epsilon(t) = -\frac{d\Phi_B}{dt} = -\sqrt{2}\omega\Phi_B^{rms}\cos(\omega t)$$

The overall sign will not be graded.

For the current, we use the extensive hints in the question to write:

$$I(t) = -\frac{\sqrt{2}\omega\Phi_B^{rms}}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t-\delta) = -\frac{\sqrt{2}\omega\Phi_B^{rms}}{\sqrt{R^2+\omega^2L^2}}\cos\left(\omega t-\tan^{-1}\frac{\omega L}{R}\right)$$

This can be rewritten in a way which will be useful for part (c):

$$I(t) = -\frac{\sqrt{2}\omega\Phi_B^{rms}}{\sqrt{R^2+\omega^2L^2}}(\cos\delta\cos(\omega t)+\sin\delta\sin(\omega t)) = -\frac{\sqrt{2}\omega\Phi_B^{rms}}{R^2+\omega^2L^2}(R\cos(\omega t)+\omega L\sin(\omega t))$$

In the last equality, we used $\tan\delta = \omega L/R$ to derive $\sin\delta = \omega L/\sqrt{R^2+\omega^2L^2}$ and $\cos\delta = R/\sqrt{R^2+\omega^2L^2}$.

All the above forms of the answer will be accepted. The overall sign will not be graded.

b. Let us forget about the metal ring, and consider a cylindrical surface at some distance z from the solenoid, with radius r and an infinitesimal height dz. The magnetic Gauss law implies that the flux through the cylinder's wall should cancel the net flux through its bases:

$$2\pi r dz B_r + \Phi_B(z+dz) - \Phi_B(z) = 0$$

Where Φ_B is the flux of the vertical magnetic field through each circular base. Dividing by dz and doing an infinitesimal amount of algebra, we get:

$$B_r = -\frac{1}{2\pi r}\cdot\frac{d\Phi_B}{dz}$$

c. The field oscillations are very slow with respect to the transit time of light through the system. Therefore, Φ_B oscillates with the same phase at all heights z, and we have:

$$B_r = -\frac{1}{\sqrt{2}\pi r}\cdot\frac{d\Phi_B^{rms}}{dz}\sin(\omega t)$$

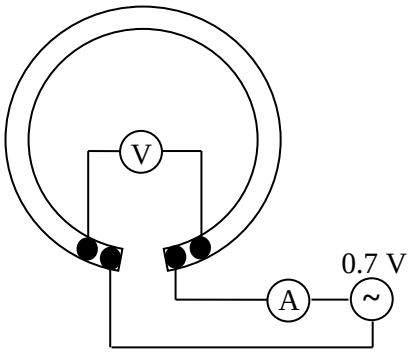
Then the momentary force reads:

$$F(t) = -2\pi r I(t)B_r(t) = -\frac{2\omega\Phi_B^{rms}}{R^2+\omega^2L^2}\cdot\frac{d\Phi_B^{rms}}{dz}\sin(\omega t)\,(R\cos(\omega t)+\omega L\sin(\omega t))$$

The time-average of $\sin(\omega t)\cos(\omega t)$ is zero, while the average of $(\sin(\omega t))^2$is $1/2$. Therefore, the time-averaged force reads:

$$\langle F\rangle = -\frac{\omega^2 L\Phi_B^{rms}}{R^2+\omega^2L^2}\cdot\frac{d\Phi_B^{rms}}{dz} = -\frac{L\epsilon^{rms}}{R^2+\omega^2L^2}\cdot\frac{d\epsilon^{rms}}{dz} = -\frac{L}{2(R^2+\omega^2L^2)}\cdot\frac{d(\epsilon^{rms})^2}{dz}$$

d. The ring's resistance is much smaller than the resistance of the electric wires and their contacts. If the voltmeter and ammeter are connected to the same two points on the ring, the measured resistance would be on the order of 0.1Ω, which is almost entirely due to the contacts. Using the multimeters on ohm-meter mode is also pointless for such resistances. Therefore, a four-terminal circuit is necessary, as shown in the figure. The contact with the ring is accomplished by snapping the "crocodiles" onto it. Note that the voltmeter's contacts must be to the inside from the current's contacts. Since no resistor is used, the power supply is effectively short-circuited, with the total resistance (and therefore, the current) mostly determined by the wires and contacts. With an optimal use of wires, a current of over 6A can be obtained. This results in a voltage of about 10mV on the ring. The ammeter's accuracy is 0.01A, while the voltmeter's accuracy is 0.1mV. This makes the voltage the primary source of measurement error, whose relative value is 1%. During the circuit's operation, the current and voltage on the ring steadily increase. Therefore, it is necessary to take the current and voltage readings simultaneously. To minimize and estimate the error in this procedure, 3 sets of measurements should be taken at slightly different values of the current (without changing the circuit). These measurements from a sample experiment are reproduced in Table 1.



I (A) ± 0.01A	V (mV) ± 0.1mV	$R_{raw} = V/I$ (mΩ) ± 1%
6.9	11.3	1.638
6.34	10.4	1.640
6.6	10.8	1.636

Table 1: Sample measurements of current and voltage on the thin ring.

In this case, we see that the statistical fluctuations are much smaller than the measurement error. In other cases, they come out similar. A reasonable estimate for the error in R_{raw} would be between 0.5% and 1%. Choosing 0.5% in our case, we write $R_{raw} = 1.638\text{m}\Omega \pm 0.008\text{m}\Omega\ (0.5\%)$.

The resistance R_{raw} is not the resistance of the entire ring, but only of the stretch between the two voltage terminals. To take this into account, we must know the distance d_{ter} between the terminals, the gap d_{gap} between the ring's ends and the average circumference πD_{thin} of the ring. The terminal distance in our sample experiment was $d_{ter} = 2.2\text{cm} \pm 0.2\text{cm}$, with the error due to the width of the contacts. The gap was $d_{gap} = 1.0\text{cm} \pm 0.05\text{cm}$, with the error due to the ruler's resolution and the width of the ring. For the arc angles associated with d_{ter} and d_{gap}, we may treat arcs as straight lines, with a negligible error of about 0.02cm.

The best way to find the average circumference is to measure the ring's outer and inner diameters with the ruler and take their average. The results are $D_{thin}^{>} = 9.60\text{cm} \pm 0.05\text{cm}$ and $D_{thin}^{<} = 9.00cm \pm 0.05\text{cm}$. Therfore, $D_{thin} = 9.30\text{cm} \pm 0.04\text{cm}\ (0.35\%)$, with the errors due to the ruler's resolution. Equivalently, a measurement with the same

accuracy can be made by placing the ring on a sheet of millimeter paper. The average circumference is now $\pi D_{thin} = 29.2\text{cm} \pm 0.1\text{cm}$ (0.35%).

Other methods, such as measuring the average diameter by taking the maximal distance between an inner point and an outer point of the ring lead to an higher error of 0.5%. Taking the inner or outer diameter *instead* of the average diameter introduces an error of about 3mm, i.e. 3%.

The true resistance of the thin ring now reads:

$$R_{thin} = R_{raw}\frac{\pi D_{thin} - d_{gap}}{\pi D_{thin} - d_{ter}} = 1.711\text{m}\Omega \qquad (1)$$

To estimate the error, we write:

$$R_{thin} \approx R_{raw}\left(1 + \frac{d_{ter} - d_{gap}}{\pi D_{thin}}\right)$$

The error of the quantity in parentheses is mainly due to $\Delta d_{ter}/(\pi D_{thin}) = 0.2\text{cm}/30\text{cm} = 0.007 = 0.7\%$. Combining this with the 0.5% error in R_{raw}, we get:

$$\frac{\Delta R_{thin}}{R_{thin}} = 0.85\%; \qquad \Delta R_{thin} = 0.015\ \text{m}\Omega$$

The distribution of sample measurement results on several different rings is consistent with this error estimate.

Neglecting to take d_{gap} into account introduces an error of $d_{gap}/(\pi D_{thin}) = 3\%$. Neglecting to take d_{ter} into account introduces an error of $d_{ter}/(\pi D_{thin}) = 6\%$. Forgetting about both and just using R_{raw} introduces an error of $(d_{ter} - d_{gap})/(\pi D_{thin}) = 3\%$.

A slightly inferior alternative to using a small d_{ter} is to connect the voltage terminals at diametrically opposite points of the ring. This decreases the measured voltage by a factor of 2, increasing its relative error by the same factor. The error in R_{raw} then becomes about 1%, slightly increasing the final error in R_{thin}.

e. As can be seen from the rings' cross-sections, the resistance R of the closed ring is smaller than R_{thin} by an order of magnitude. This makes a naïve 2-terminal measurement even more hopeless. A 4-terminal measurement as in part (d) is possible, but will result in a large error of about 5% due to the voltmeter's resolution. Furthermore, for the closed ring the inductive impedance ωL is no longer negligible, and will introduce a systematic error of about 3%.

The optimal solution is to use the fact that the rings are made of the same material, and deduce R from the rings' geometries and the accurately measured R_{thin}:

$$R = R_{thin}\frac{\pi D_{closed}}{\pi D_{thin} - d_{gap}} \cdot \frac{A_{thin}}{A_{closed}}$$

where A stands for the cross-section area. The average diameter D_{closed} of the closed ring can be found as in part (d), with the results $D^{>}_{closed} = 10.05\text{cm} \pm 0.05\text{cm}$ and $D^{<}_{closed} = 7.60\text{cm} \pm 0.05\text{cm}$. Therefore, $D_{closed} = 8.82\text{cm} \pm$

0.03cm. In this case, using the inner or outer diameter instead of the average one introduces a large error of 1.2cm, i.e. over 10%.

Measuring A_{thin} directly also introduces large errors. Measuring the thin ring's thickness and height with the ruler introduces an error of $0.5\text{mm}/3\text{mm} = 15\%$ for each dimension. Multiplying by $\sqrt{2}$, this implies an error of 25% in the area. A student may also try to measure the ring's dimensions using the screw attached to the solenoid. Then the measurement error for each dimension decreases to about 1/16 of a screw step, i.e. $0.09\text{mm}/3\text{mm} = 3\%$, which implies a 4% error in the area.

The solution is to weigh the rings using the digital scale. We then have:

$$\frac{A_{thin}}{A_{closed}} = \frac{m_{thin}}{m_{closed}} \cdot \frac{\pi D_{closed}}{\pi D_{thin} - d_{gap}}$$

$$R = R_{thin}\left(\frac{\pi D_{closed}}{\pi D_{thin} - d_{gap}}\right)^2 \cdot \frac{m_{thin}}{m_{closed}} = 0.153\text{m}\Omega$$

where we used the values $m_{thin} = 4.50\text{g} \pm 0.02\text{g}$ and $m_{closed} = 47.70\text{g} \pm 0.02\text{g}$ from our sample measurement. The measurement error for the masses depends on environmental noise, and we use 0.02g as a representative value. The rings in different experimental sets have slightly different masses, with a deviation of about 1%, which can be distinguished at the scale's level of sensitivity. Therefore, different students will measure slightly different values.

The dominant error in the mass ratio is $\Delta m_{thin}/m_{thin} = 0.5\%$. The error in πD_{closed} is 0.35%, as in part (d). After taking the square, this doubles to 0.7%. Examining eq. (1), we see that the relative error in $R_{thin}/(\pi D_{thin} - d_{gap})^2$ is the same as in R_{thin}, since the $(\pi D_{thin} - d_{gap})$ merely moves from the numerator to the denominator. The error in $R_{thin}/(\pi D_{thin} - d_{gap})^2$ is therefore 1%. Combining these three error sources, we have:

$$\frac{\Delta R}{R} = \sqrt{0.0085^2 + 0.007^2 + 0.005^2} = 1.2\%; \qquad \Delta R = 0.002\text{m}\Omega$$

The distribution of sample measurement results on several different rings is consistent with this error estimate; the value $R = 0.153\text{m}\Omega$ cited above is near the bottom of the distribution.

A student who neglects d_{gap} will get a systematic error of 6%. A student who neglects the difference between D_{closed} and D_{thin} will get a systematic error of 10%. A student who neglects both, i.e. just uses $R = R_{thin}(m_{thin}/m_{closed})$, will get an error of 4%. These errors are halved if the student makes them for just one of the two factors of $\pi D_{closed}/(\pi D_{thin} - d_{gap})$.

In the above, we effectively treated the closed ring as a rectangle with length D_{closed}. It's easy to see that this introduces no errors in the mass calculation. Indeed, the precise formula for the ring's volume reads:

$$V = \pi a\left(\left(\frac{D_{closed} + w}{2}\right)^2 - \left(\frac{D_{closed} - w}{2}\right)^2\right) = \pi D_{closed} w a$$

where a is the ring's height, and w is its width. This is the same formula as in the rectangular approximation. Some students may use this derivation in their solution.

The exact resistance calculation for a broad circular ring is more difficult, and reveals that the relative error from the rectangular approximation is $w^2/(3D_{closed}^2) = 0.6\%$. This analysis is not expected from the students, and the resulting error can be neglected with respect to the overall error of 1.5%. For completeness, we include the derivation of the exact formula:

$$\frac{1}{R} = \frac{1}{\rho}\int_{(D_{closed}-w)/2}^{(D_{closed}+w)/2}\frac{adr}{2\pi r} = \frac{a}{2\pi\rho}\ln\frac{D_{closed}+w}{D_{closed}-w} \approx \frac{wa}{\pi D_{closed}\rho}\left(1+\frac{1}{3}\left(\frac{w}{D_{closed}}\right)^2\right)$$

where ρ is the material resistivity.

f. The student can vary z by using the screw to raise and lower the solenoid. The most precise way to measure z is simply to count the number of screw steps. The error is then $\Delta z = h/16 = 0.09\text{mm}$, where $h = 1.41\text{mm}$ is the screw step. If z is measured with the ruler, the error becomes $\Delta z = 0.5\text{mm}$, due to the ruler's resolution. A convenient point to define as $z = 0$ is when the solenoid touches the ring from above, and the screw's handle is in some fixed orientation. This point should be reproducible, either visually or by counting screw steps, in order to keep a consistent record of distances with the force measurements in the next part.

In anticipation of the force measurements, the student should place the scale under the solenoid, place the polystyrene block on the scale, and place the ring on the polystyrene block. The polystyrene block is important because it is an insulator, while the scale's platform is metallic and may alter the EMF-measuring circuit. It is also important for the quality of the force measurements, as explained below.

It is always best to start measuring from a small distance, because then the ring can be aligned with the solenoid's axis more easily. A reasonable range of z would be from $z = 0$ (near-contact with the solenoid) to $z = 5\text{cm}$. A reasonable resolution is one screw step. It can be made coarser towards large z, when the EMF variations become smaller.

The EMF can be measured by connecting the ends of the broad open ring to the voltmeter. We wish to measure the magnetic flux through the ring, and not through the rest of the circuit. To make sure that this is the case, the wires from the ring should be twisted into a braid. The EMF decreases with distance from about $\epsilon_{rms} = 22\text{mV}$ to zero, reaching about $\epsilon_{rms} = 5\text{mV}$ at $z = 5\text{cm}$. The measurement error is $\Delta\epsilon_{rms} = 0.1\text{mV}$. Sample measurement results are presented in Table 2. A plot of the measurements (with a trend line for part (h)) is presented in Graph 1.

g. The student can vary z as in part (f), this time measuring the force on the closed ring using the digital scale. Care should be taken to use the same zero point for z as in part (f). It is convenient, though not necessary, to measure the force at the exact same points where the EMF was measured previously.

Again, the ring should be placed on the polystyrene block, rather than directly on the scale. There are two reasons for this in the context of force measurements. First, the metallic parts of the scale also react to the solenoid's magnetic field. Therefore, the solenoid must be kept at a distance above the scale, to eliminate a direct effect on the scale's reading. To observe this effect and its successful elimination, the student may turn on the current in the solenoid without a ring resting on the scale, and check whether the scale's reading changes. The second reason to use the polystyrene block is the small area of the scale; if the ring rests on the scale directly, it's difficult to ensure that some of its weight doesn't fall on the scale's lid or other supporting surfaces. Finally, one must make sure that the solenoid isn't in direct contact with the ring or the solenoid block, so that its weight doesn't fall on the scale.

It is convenient to turn on the scale, or to press the Tare button, with the ring and the block resting on the scale with no current in the solenoid. We can then measure the magnetic force directly. Otherwise, the student must manually subtract the scale's reading at zero current from all of his force values.

For values of z from $z = 0$ to $z = 5\text{cm}$, the force decreases from about $\langle F\rangle = 8\text{gf}$ (grams-force) to about $\langle F\rangle = 0.25\text{gf}$. The measurement error depends on environmental noise. A representative value is $\Delta\langle F\rangle = 0.02\text{gf}$. Sample measurement results are presented in Table 2.

h. The derivative should be found from discrete differences between pairs of points, situated symmetrically around the point we are interested in. It is better to find the derivative $d\epsilon_{rms}/dz$, and then multiply it by $2\epsilon_{rms}$, than to find the derivative $d\epsilon_{rms}^2/dz$ directly. This is for two reasons. First, taking the square amplifies the errors in the discrete point differences. Second, if the student chooses a graphical method (see below), it is more convenient to use the graph of $\epsilon_{rms}(z)$: it was already drawn in part (f), and its points are more evenly distributed than the points on a graph of $\epsilon_{rms}^2(z)$.

We will now discuss two distinct methods for choosing the discrete differences for $d\epsilon_{rms}/dz$. When used properly, the two give equally good results.

Numerical method:
One method to find the derivative $d\epsilon_{rms}/dz$ is simply to take differences between measured values of ϵ_{rms}. The intervals at which the differences are taken must be carefully chosen. A small interval will give a large error in the slope, due to the statistical scatter of the measured points. On the other hand, a large interval may result in too much smearing, so that we're not capturing the local slope. An optimal interval is about 6 screw steps, i.e. about $3h = 4\text{mm}$ to each side from the point of interest.

Graphical method:
Another method is to draw a smooth trend line through the measured points, and then to use differences between points on this trend line rather than the measured values. If the measurements were carried out properly, the trend line will deviate by only $0.5\text{mm} - 1\text{mm}$ from the measured points on the graph paper. An example worked out by a hapless theoretician is shown on Graph 1. The thick line is a consequence of the trial-and-error process of sketching the best line. The discrete intervals for the derivative can now be chosen smaller than with the numerical method, since the statistical scatter is smoothed out. In particular, taking one screw step to each side from the point of interest is now good enough. One should not choose much smaller intervals, due to the limited resolution of the graph paper. A potential advantage of the graphical method is that it's not tied to the exact z values where the EMF was measured. This is helpful if the EMF and the force were not measured at the exact same heights.

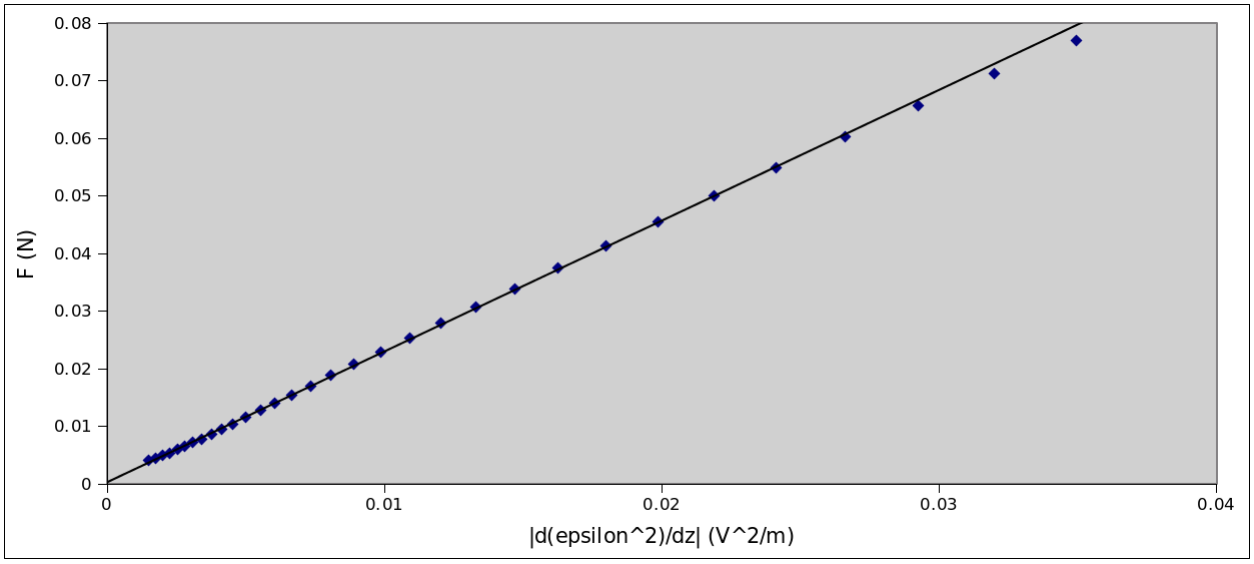
An inferior graphical method is to draw tangents to the curve $\epsilon_{rms}(z)$, and calculate the slopes of these tangents. In practice, it is very difficult to identify a tangent line visually, and using this method produces large deviations, on the order of 20%, in the analysis of part (i).

Sample values of $d\epsilon_{rms}/dz$ and $d\epsilon_{rms}^2/dz$ are provided in Table 2.

i. Following parts (c) and (h), the student should draw a linear graph of $\langle F\rangle$ as a function of $d\epsilon_{rms}^2/dz$. The graph should pass through the origin, and its slope equals:

$$k = \frac{L}{2(R^2 + \omega^2 L^2)} \qquad (2)$$

If very small distances are included in the graph (under about 6mm between the ring and the solenoid's edge), the corresponding points will deviate from linearity, with a visible decrease in the slope. This can be seen in the computerized plot below:



This effect results from attraction between the ring and the solenoid's iron core.

In the manual graph from our sample experiment (Graph 2), the non-linearity isn't observed, and the slope comes out $k = 2.29\text{Nm/V}^2 \pm 0.04\text{Nm/V}^2$ (2%). To extract L, we must solve the quadratic equation:

$$2\omega^2 k L^2 - L + 2kR^2 = 0$$

The two roots are:

$$L = \frac{1 \pm \sqrt{1 - 16\omega^2 k^2 R^2}}{4\omega^2 k}$$

where we must use $\omega = 2\pi \cdot 50\text{Hz} \approx 314\text{Hz}$.

The student can find, either analytically or by substitution, that only the smaller root satisfies $\omega L < R$. In our sample experiment, the result reads:

$$L = \frac{1 - \sqrt{1 - 16\omega^2 k^2 R^2}}{4\omega^2 k} = 1.13 \cdot 10^{-7}\text{H} = 0.113\mu\text{H}$$

If the student hasn't done so before this point, he will need convert the force units from grams-force to Newtons using the provided value of g.

We find that the ratio $\omega L/R$ is 0.23. Therefore, for the error estimation, we can write eq. (2) as:

$$k = \frac{L}{2R^2} \quad \Rightarrow \quad L = 2kR^2 \qquad (3)$$

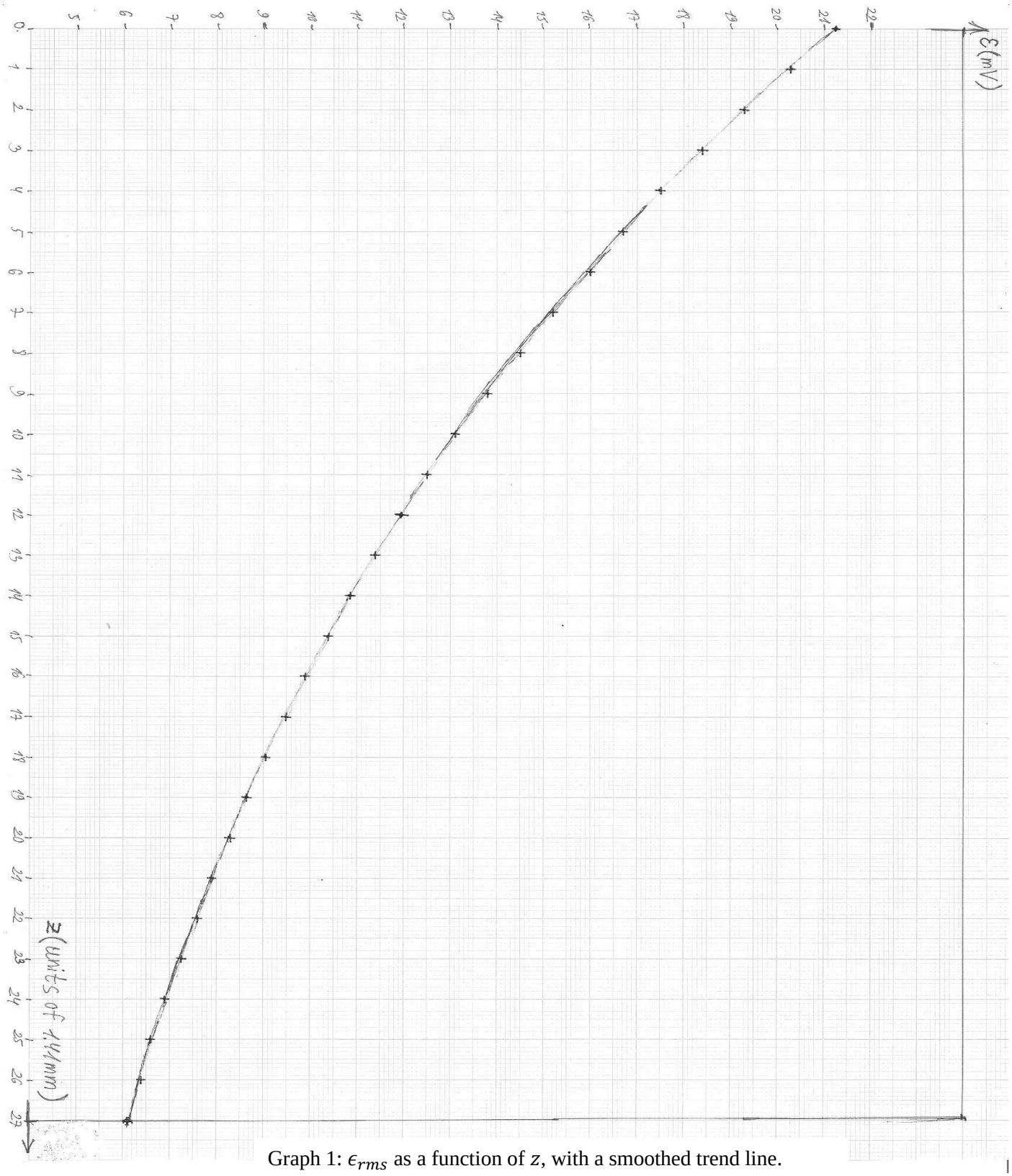
As we can see, the delicate error considerations in R are now even more important, since it appears squared. Collecting the relative errors in k and R, we have:

$$\frac{\Delta L}{L} = \sqrt{0.02^2 + (2 \cdot 0.012)^2} = 0.03 = 3\%; \qquad \Delta L = 0.003\mu H$$

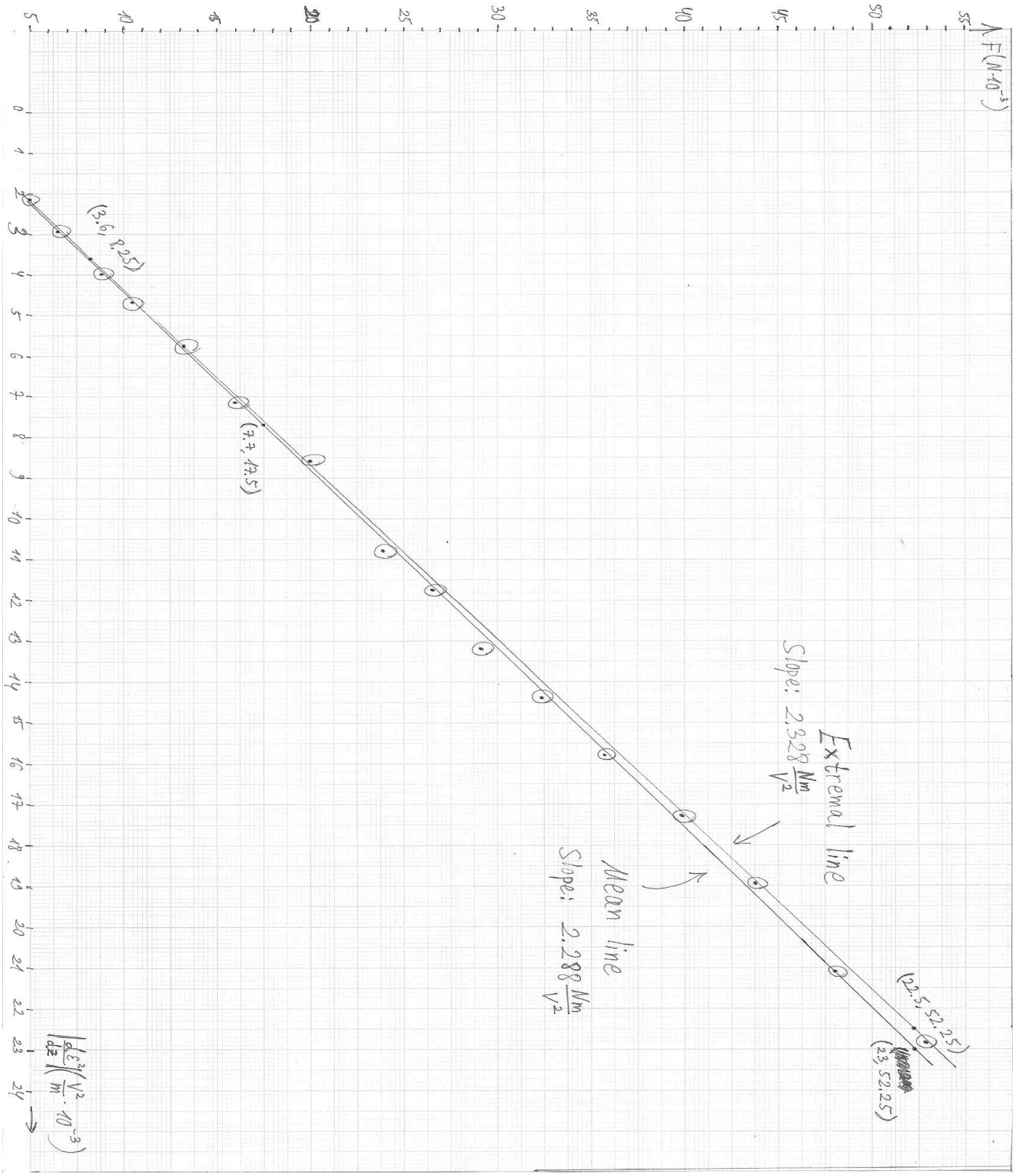
The scatter of results from sample experiments is consistent with this error estimation.
A student who neglects $(\omega L/R)^2$ all along and uses eq. (3) to find the *value* of L will introduce a systematic error of 5%.

n (screw steps) ± 0.1	z (mm) ± 0.15mm	ϵ_{rms} (mV) ± 0.1mV	$\langle F \rangle$ (gf) ± 0.02gf	$\langle F \rangle$ (N) ± 0.0002N	$\lvert d\epsilon_{rms}/dz \rvert$ (V/m)	$\lvert d\epsilon_{rms}^2/dz \rvert$ (V^2/m)
0	0	21.25	7.08	0.06938		
1	1.41	20.3	6.5	0.06370		
2	2.82	19.3	5.93	0.05811		
3	4.23	18.4	5.4	0.05292	0.6206	0.02284
4	5.64	17.5	4.9	0.04802	0.6028	0.02110
5	7.05	16.7	4.47	0.04381	0.5674	0.01895
6	8.46	16	4.07	0.03989	0.5437	0.01731
7	9.87	15.2	3.65	0.03577	0.5201	0.01581
8	11.28	14.5	3.3	0.03234	0.4965	0.01440
9	12.69	13.8	2.97	0.02911	0.4787	0.01321
10	14.1	13.1	2.71	0.02656	0.4492	0.01177
11	15.51	12.5	2.44	0.02391	0.4314	0.01079
12	16.92	11.95	2.2	0.02156	0.4019	0.00961
13	18.33	11.4	2.045	0.02004	0.3783	0.00862
14	19.74	10.85	1.83	0.01793	0.3546	0.00770
15	21.15	10.4	1.64	0.01607	0.3428	0.00713
16	22.56	9.9	1.45	0.01421	0.3251	0.00644
17	23.97	9.5	1.35	0.01323	0.3014	0.00573
18	25.38	9.05	1.2	0.01176	0.2955	0.00535
19	26.79	8.65	1.07	0.01049	0.2719	0.00470
20	28.2	8.3	1	0.00980	0.2660	0.00442
21	29.61	7.9	0.905	0.00887	0.2541	0.00402
22	31.02	7.6	0.815	0.00799	0.2423	0.00368
23	32.43	7.25	0.74	0.00725	0.2246	0.00326
24	33.84	6.9	0.67	0.00657	0.2128	0.00294
25	35.25	6.6	0.61	0.00598	0.2009	0.00265
26	36.66	6.4	0.56	0.00549	0.1950	0.00250
27	38.07	6.1	0.52	0.00510	0.1773	0.00216
28	39.48	5.9	0.47	0.00461		
29	40.89	5.6	0.42	0.00412		
30	42.3	5.4	0.36	0.00353		

Table 2: Sample EMF and force measurements and derivative values



Graph 1: ϵ_{rms} as a function of z, with a smoothed trend line.



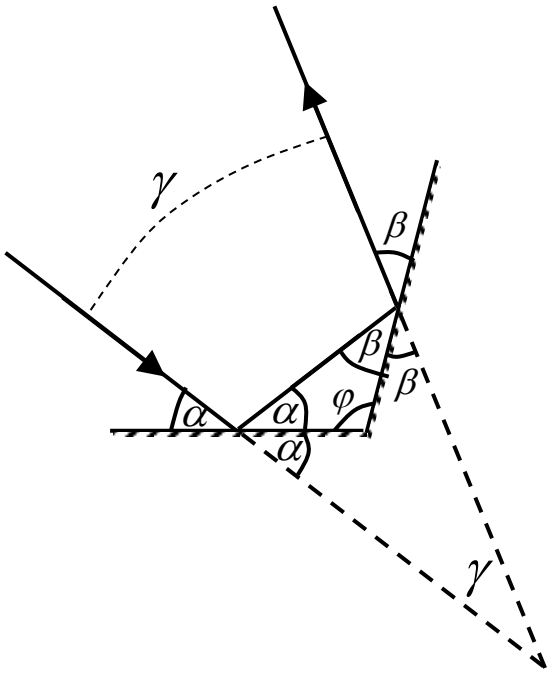
Graph 2: $\langle F \rangle$ as a function of $|d\epsilon_{rms}^2/dz|$, with linear trend lines.

Experimental Question 2: An Optical "Black Box"

SOLUTION

a. Consider an incoming light beam at angle α to the first mirror (see figure). The angle to the second mirror is given by $\beta = \pi - \varphi - \alpha$. Thus, the total deflection angle is $2\alpha + 2\beta = 2(\pi - \varphi)$. In particular, it is independent of α. The angle γ is then given by:

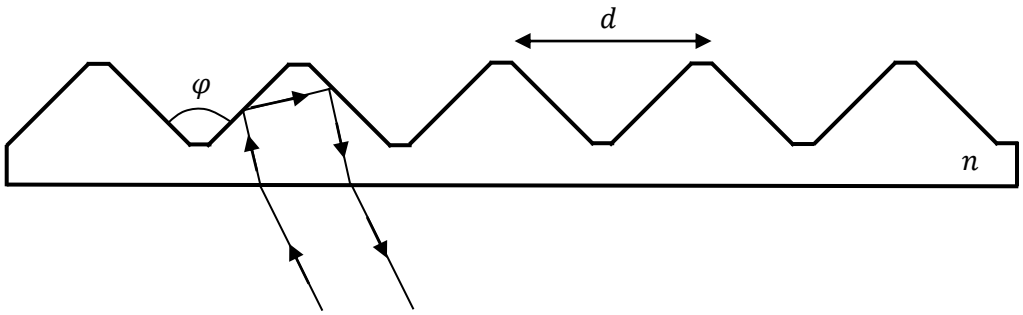
$$\gamma = \pi - 2(\pi - \varphi) = 2\varphi - \pi$$



b. The correct option is A. Options C and D are ruled out, because both the transmission and reflection patterns possess left-right symmetry. Option B is ruled out, because it implies strong forward transmission at perpendicular illumination; instead, strong transmission is observed only to the sides, as described in question (d).

c. To determine φ, we must observe the reflection pattern, using the setup suggested in the question. It is important to place the flashlight into the hole in the screen; otherwise, the observed pattern will be smeared due to the sample's non-zero size.

A strong reflection pattern is observed when the light enters the sample from its flat side, is reflected twice from the two slanted faces, and exits again from the flat side. This optical path results in total internal reflection, producing a much brighter reflection pattern than the pattern for light coming from the "toothed" side.



The observed pattern is a colorful band of interference fringes, gradually fading to the sides, without well-defined edges. As we rotate the sample, it can be seen that the pattern widens and narrows, while its center remains stationary, directly on top of the flashlight's center. While the color sequence is complicated, it is visibly symmetric around the center, which appears white; these observations are the key to identifying the center precisely.



The center of the pattern corresponds to the optical path due to geometric optics. From part (a), we see that such a zero reflection angle corresponds to an angle $\varphi = \pi/2$ between the two reflecting surfaces. This conclusion is supported by two other qualitative observations. First, if the reflection angle were nonzero, light from different areas of the sample would reach different points on the screen, due to the sample's non-zero size. This would have lead to a smeared reflection pattern, like the one obtained when the flashlight does not coincide with the screen. Second, we see that the location of the pattern's center doesn't depend on the sample's orientation. Without refraction at the flat boundary, this would have been the case for any reflection angle. However, taking the refraction into account, the angle between the incoming and outgoing rays would have to be a function of the incidence angle, *except* for the case $\varphi = \pi/2$, for which the angle remains zero after refraction (see figure above).

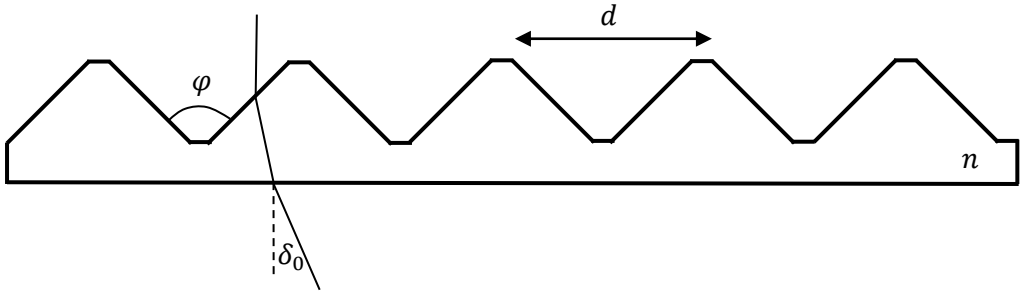
The center of the interference pattern can be identified with an error of $\Delta x = 0.5\text{mm} - 1\text{mm}$ for the optimal observation distance $y \approx 50\text{cm}$ between the flashlight and the sample. This results in an error of at most 0.002 radians in the angle between the incoming and outgoing rays. Examining the result of part (a), we see that this error should be halved to give the error in φ. Though it is not required, a bright student will also divide the error by the refraction index n, which can be estimated as ~ 1.5. This is because the refraction amplifies angular deviations as the light leaves the sample. We conclude that the error in φ is $\Delta\varphi = 7 \cdot 10^{-4}\text{rad} = 0.04^{\text{o}}$. This will be negligible with respect to all other errors in the experiment.

A common mistake will be to take the edge of the central white stripe, or the edge of the entire pattern, as representing φ instead of the pattern's center. For the edge of the central white stripe (about $x = 5\text{mm}$ at $y = 50\text{cm}$), this would result in a $0.15^{\text{o}} - 0.3^{\text{o}}$ deviation (depending on whether n is taken into account). For the edge of the pattern (about $x = 3\text{cm} - 5\text{cm}$ at $y = 50\text{cm}$), this would result in a $1^{\text{o}} - 3^{\text{o}}$ deviation.

Similarly, a student may decide that the edge of the central white stripe or the edge of the pattern correspond to the error $\Delta\varphi$. This will lead to an over-estimation of the error as $0.15^{\text{o}} - 0.3^{\text{o}}$ or $1^{\text{o}} - 3^{\text{o}}$, respectively.

d. The correct option is C. An interference pattern will always have larger deflection angles for larger wavelengths. Therefore, the pattern in Figure 7 cannot result from interference, and must result from the dependence of n on the wavelength. On the other hand, the pattern in Figure 6 has the correct order of colors for an interference pattern. It must in fact be an interference pattern, because the sample's structure as depicted in part (b) is rather simple and periodic, and cannot produce a large number of nearby refraction angles.

e. The phenomenon described in this part is observed when the sample is illuminated from its "toothed" side. Transmission from the other side is very weak due to total internal reflection.



The deflection angle δ_0 can be seen qualitatively by projecting the transmission pattern onto the screen. However, due to the sample's nonzero size, this pattern is too smeared to enable precise measurement. A much sharper pattern is observed when looking directly through the sample, as suggested in the question. The challenge is then to measure angles that are seen directly with the eye, rather than on a screen.

For this purpose, the student is provided with the mobile wooden stake on a bench. The bench should be placed perpendicularly to the line of sight between the flashlight and sample. The stake should be positioned at two points: point A directly along the line of sight between the flashlight and sample, and point B where the stake coincides with the desired band of deflected light along the student's line of sight. The two measurements are the distance $x = AB$ between these two points and the distance y between the sample and the center of the bench. The desired deflection angle is then given by:

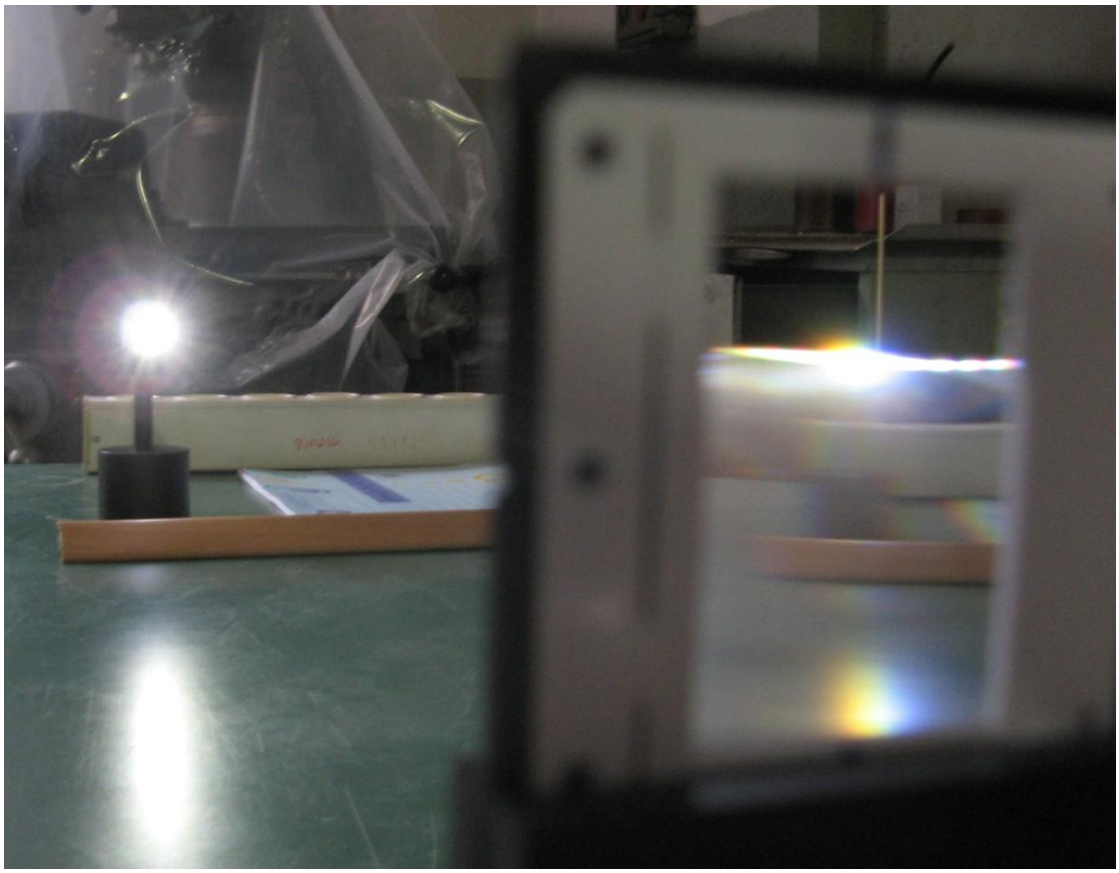
$$\delta_0 = \tan^{-1}\frac{x}{y}$$

To properly establish point A, some alignment issues must be solved. The long bench can be quite accurately placed in parallel to one of the desk's sides. The flashlight can then be placed near the bench, in perpendicular to it. The sample can also be aligned in parallel to the desk's side, by visual reference to the orientation of its long handle. The bench should lie in front of the flashlight, rather than behind it. This has two advantages. First, it allows for a greater distance between the flashlight and sample, minimizing the harm from the sample's nonzero size. Second, it allows us to align the stake directly in front of the flashlight quite accurately. The alignment of the stake and flashlight can be checked and corrected by observing the stake's shadow on the desk. Finally, we must place the sample precisely in front of the flashlight, i.e. without a sideways offset. One way to do this is to place both the flashlight and sample at the edge of the desk. A less accurate way is to measure their distances from the edge using the tape measure. In any case, there are several ways to check the alignment without reference to the desk:

1. Ensuring that the sample lies at the center of the stake's shadow from the flashlight.
2. Looking at the flashlight through the sample's frame, and making sure that the flashlight's center and the stake are aligned together with the frame's center.

3. Placing a sheet of paper behind the sample, and making sure that the stake's shadow on the paper is at the center of the shadow from the sample's frame.

To establish point B, it is best to look through the sample not from a zero distance, but from a distance large enough to focus the eye on the sample's frame. One should then try to align three points along the line of sight: the center of the sample's frame (marked with a blue line), the desired band of deflected light, and the stake. It is convenient first to align the frame's center with the deflected light, and then check in which direction the stake should be moved to align with them. Within 3-5 such iterations, point B can be established with an accuracy of about $\pm 2\text{mm}$.



Once the points A and B are physically established, the distance between them has to be measured. One way is to record each position of the stake with a pen mark on the bench, and then measure the distance between the two markings. It's difficult and unnecessary to mark the position of the stake itself – one can mark the position of the edge of the stake's holder. Another way, which doesn't require any markings, is to align the edge of the stake's holder at point A with the edge of the bench. Then when the stake is at point B, the distance x can be measured directly. In any case, while moving the stake between points A and B, the student must take care not to move the bench.

The relative accuracy of the measurement increases with the distance between the flashlight and bench and the sample. Optimally, most of the desk's 1-meter length should be used for the distance y. Then x will take up most of the desk's width. The price for these large distances is that the bench is not within arm's reach from the sample, and the student will have to get up for each adjustment of the stake's position.

The cylindrical base of the sample holder makes it difficult to measure the distance y accurately. The solution is to place the measuring tape perpendicularly to the bench or screen, and in tangent to the cylindrical base. Repeated measurements reveal that this reduces the error to 1mm.

We now present the results of a sample measurement:

$$x = 471\text{mm} \pm 2\text{mm}, \qquad y = 775\text{mm} \pm 1\text{mm}$$

From these we derive the deflection angle $\delta_0 = \tan^{-1}(x/y) = 31.3^{\text{o}}$. The error in δ_0 as derived from the measurement errors reads:

$$\Delta\delta_0 = \sqrt{\left(\frac{\Delta x/y}{1+x^2/y^2}\right)^2 + \left(\frac{x\Delta y/y^2}{1+x^2/y^2}\right)^2} = 0.002\text{rad} = 0.1^{\text{o}} \qquad (1)$$

One can also use the simpler formula $\Delta\delta_0 \approx \Delta(x/y)$ to arrive at a similar result. The value $\Delta x = 2\text{mm}$ leading to this estimate was obtained from repeated attempts to locate point B without changing the overall alignment. A student who repeats the alignment process from the beginning and compares his results will likely get a larger error estimate, up to $\Delta\delta_0 = 0.3^{\text{o}}$. The source of these deviations is the difficulty in arranging perpendicular incidence of the light onto the sample. This error is difficult to quantify in any way other than repeating the alignment process. An additional source of error is the subjective definition of the "blue end of the visible spectrum". Sample measurements were performed by different individuals, and the scatter of their results is taken into account in the Marking Scheme.

Instead of taking the distance between the source and the deflected light band, a student may take the distance between the two symmetric deflected bands on either side of the source. This carries two disadvantages: the size of the desk constrains the perpendicular distance y in this case to about 40cm, and the difficulties with locating point B are now encountered twice.

Another inferior alternative is to use the screen's edges or center instead of the wooden stake. Then the distance x is fixed by the screen's width, and y becomes the controlled variable. This method leads to higher errors (about 0.5^{o}), for two reasons. First, the stake is helpful in the alignment of the different elements, as described above. Second, it is difficult to fine-tune the distance y without spoiling the alignment.

f. In this part, after the initial alignment and before measuring the deflection angle, the student should gently rotate the sample while looking through it, until he identifies the orientation at which the deflection angle is minimal. Unlike the deflection of perpendicular light in part (e), this minimal deflection angle can be observed from both sides of the sample. When the light is incident on the "toothed" side, as in part (e), the minimum is obtained not far from perpendicular incidence. As a result, δ_{min} is only slightly smaller than δ_0.

The measurement process is very similar to that in part (e), except that there's no need to align the sample for perpendicular incidence. The student may keep the rest of the alignment from part (e), and use the same distance y between the sample and the wooden bench. The results of a sample measurement read:

$$x = 461\text{mm} \pm 2\text{mm}, \qquad y = 775\text{mm} \pm 1\text{mm}$$

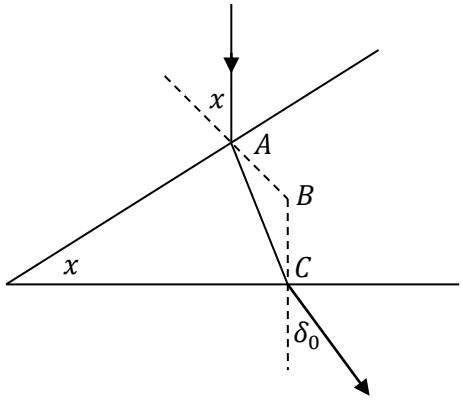
From which we derive:

$$\delta_{min} = \tan^{-1}(x/y) = 30.75^{o} \pm 0.1^{o} \qquad (2)$$

In this case, the naïve error estimation of eq. (1) corresponds to the actual scatter of results from repeated alignments and measurements. This is because the error from the perpendicular alignment is gone. We still have an error in finding the sample orientation which leads to the minimal deflection. However, this error has very little impact on the measurement of δ_{min} itself, since the deflection angle is stationary around the minimum.

g. Solution in terms of δ_0:

Consider a vertical light ray entering a horizontal prism with opening angle x (see figure). We have:



$$\sin \sphericalangle BAC = \frac{\sin x}{n}$$
$$\sphericalangle ABC = \pi - x$$
$$\sphericalangle BCA = \pi - \sphericalangle BAC - \sphericalangle ABC = x - \sphericalangle BAC$$
$$\sin \delta_0 = n \sin \sphericalangle BCA = n \sin(x - \sphericalangle BAC) =$$
$$= n(\sin x \cos \sphericalangle BAC - \cos x \sin \sphericalangle BAC) =$$
$$= n\left(\sin x \sqrt{1 - \left(\frac{\sin x}{n}\right)^2} - \cos x \frac{\sin x}{n}\right) =$$
$$= \sin x \left(\sqrt{n^2 - (\sin x)^2} - \cos x\right)$$

Solving for n, we get:

$$n = \sqrt{(\sin x)^2 + \left(\cos x + \frac{\sin \delta_0}{\sin x}\right)^2}$$

In our case, the appropriate prism angle is $x = (\pi - \varphi)/2$, which gives:

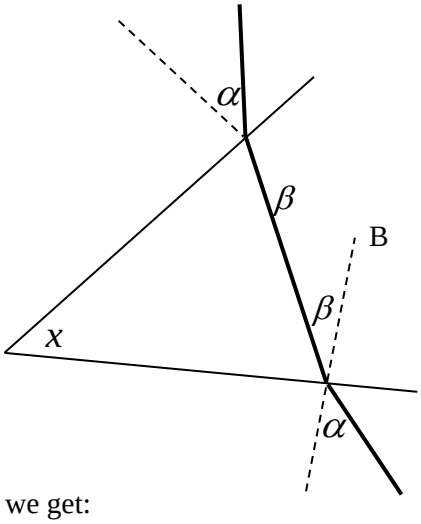
$$n = \sqrt{(\cos(\varphi/2))^2 + \left(\sin(\varphi/2) + \frac{\sin \delta_0}{\cos(\varphi/2)}\right)^2}$$

A student that plugs in the value $\varphi = \pi/2$ will get:

$$n = \sqrt{\frac{1 + (1 + 2\sin \delta_0)^2}{2}}$$

Solution in terms of δ_{min}:

Using the uniqueness of the minimal deflection angle δ_{min} and the reversibility of light rays, we conclude that minimal deflection occurs symmetrically with respect to the prism angle:



$$\sphericalangle B = \pi - x$$
$$\beta = \frac{\pi - \sphericalangle B}{2} = \frac{x}{2}$$
$$\delta_{min} = 2(\alpha - \beta) = 2\alpha - x$$
$$n \sin\frac{x}{2} = n \sin\beta = \sin\alpha = \sin\frac{x + \delta_{min}}{2}$$

Solving for n, we get:

$$n = \frac{\sin\frac{x + \delta_{min}}{2}}{\sin\frac{x}{2}}$$

In our case, the appropriate prism angle is $x = (\pi - \varphi)/2$, which gives:

$$n = \frac{\sin\frac{\pi - \varphi + 2\delta_{min}}{4}}{\sin\frac{\pi - \varphi}{4}}$$

A student that plugs in the value $\varphi = \pi/2$ will get:

$$n = \frac{\sin\frac{\pi + 4\delta_{min}}{8}}{\sin\frac{\pi}{8}} \qquad (3)$$

h. The refraction index n_v can be found either from δ_0 or from δ_{min}, using the formulae from part (g). As discussed in parts (e) and (f), the measurement of δ_{min} is the more accurate one. Therefore, its accuracy is the basis for the full credit in the Marking Scheme for this part. Using the results (2) and (3), we find for our sample experiment:

$$n_v = 1.604$$

The error reads:

$$\Delta n_v = \frac{\cos\frac{\pi + 4\delta_{min}}{8}}{2\sin\frac{\pi}{8}} \Delta\delta_{min} = 0.002 \qquad (4)$$

i. The measurement process here is the standard one for a diffraction grating. The pattern is projected onto the screen. The measured quantities are the distance y between the sample and the screen, the distances x on the screen between the fringes and the pattern's center, and the fringe number m. The angles θ for the different fringes are then found as $\theta = \tan^{-1}(x/y)$. It is important that x is measured from the center, since the relation $d \sin\theta = m\lambda$ is linear in $\sin\theta$ and not in $x \sim \tan\theta$. An offset in m is less problematic, since it will not affect the slope of the linear graph in part (j).

The alignment process is more straightforward than with the white light. The laser beam, observed as scattered light from the screen or from the sample, can be used as a reference line instead of the stake's shadow. The sample's nonzero size no longer leads to measurement uncertainties, since the laser beam illuminates only a small portion of it. Instead, we are constrained by the width of the beam, which widens noticeably with distance. Since the laser pointer is not quite horizontal, the beam may come out at a slight vertical slope, causing it to miss the sample. To fix this, the student may rotate the laser around its axis.

A diffraction pattern can be observed in three different configurations:

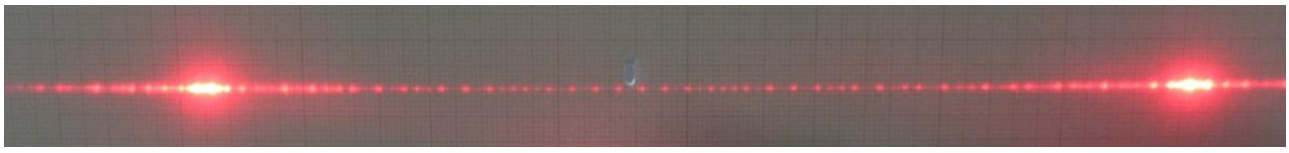
1. Reflection with the light entering the sample from the flat side, as in part (c).
2. Transmission with the light entering the sample from the "toothed" side, as in parts (d)-(e).
3. Transmission with the light entering the sample from the flat side.

All three configurations produce fringes at the same angles $\theta(m)$, but otherwise they look quite different. Each configuration has its advantages and disadvantages for our measurement purposes. The best choice among the three may depend on the amount of ambient light in the room.

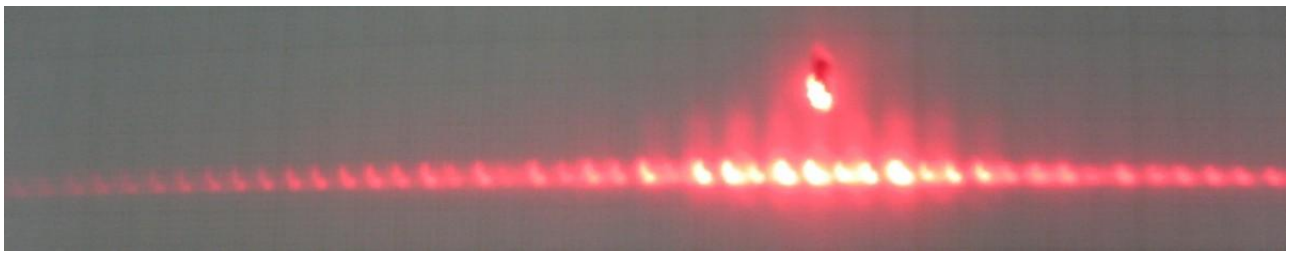
Configuration 3 gives a wide pattern of dim but well-defined fringes. They result from the interference of light passing through the narrow flattened "peaks" and "valleys" of the sample's toothed side (the light incident on the slanted faces of the "teeth" undergoes total internal reflection). Note: the observed patterns are sharper than in the photos below, which were taken by a theoretician.



Configuration 2 gives a similar pattern from the "peaks" and "valleys", superimposed on two much brighter narrow interference patterns from the slanted faces. These bright patterns are centered around the geometric deflection angle δ_0, one to the right and one to the left. They are bright because more light passes through the slanted faces, and they are narrow because they are multiplied by the narrow diffraction pattern off of each slanted face, whose width is the same order of magnitude as the spacing d. The condition $d\sin\theta = m\lambda$ for a fringe at angle θ is the same for the "peaks" and "valleys" and for the slanted faces.



Configuration 1 (reflection) gives a noisier interference pattern, with broader fringes. This is because the laser beam travels the distance to the screen twice (back and forth), and broadens in the process. The central fringes are very bright – most of the light is reflected off of both slanted faces directly backwards, as discussed in the solution to part (c). These fringes are even brighter than the bright patterns in Configuration 2, because in this case the beam doesn't split in two. This bright region of the reflected pattern is narrow, again due to the narrow diffraction pattern off of the broad slanted faces. Further to the sides, the dim interference pattern from the "peaks" and "valleys" becomes dominant. It is dimmer than in the transmission patterns, and fades away quickly, because it arises from weak reflection, as opposed to the total internal reflection from the slanted faces.



The dim interference pattern from the narrow "peaks" and "valleys" on the sample displays an alternating sequence of brighter and dimmer fringes, as described in the question text. If all the fringes are taken into account, their spacing corresponds to the sample's spacing d. The twice larger spacing between the *brighter* fringes corresponds to the spacing $d/2$ between each "peaks" and the adjacent "valley". In other words, the dimmer fringes are due to destructive interference between a peak and its adjacent valley. Towards larger θ, the effect becomes less simple than just a factor-of-2, because then the vertical distance between the "peaks" and "valleys" must also be taken into account. As the sample is rotated, the brighter and dimmer fringes interchange, and at some orientations become equal in magnitude. This phenomenon is more pronounced in the transmission pattern, i.e. Configurations 2 and 3. In the reflection pattern, i.e. Configuration 1, it is pronounced towards the edges, where the light from the "peaks" and "valleys" becomes dominant. The central fringes in Configuration 1 remain fixed up to small changes in brightness, and their spacing corresponds to d. The bright side fringes in Configuration 2 vary noticeably in brightness, due to the diffraction pattern from each slanted face.

It is likely that some of the dimmer fringes in the alternating sequence will not be visible. This results in a doubled, and sometimes tripled, spacing between the visible fringes. By rotating the sample and observing the alternating pattern, the student should learn to identify such doubled spacing, and take it into account when recording the fringe numbers m.

In the reflection setup, rotation of the sample reveals a bright fringe moving across the screen. This fringe is of similar brightness to the central fringes of the interference pattern. It arises from light directly reflected from the sample's flat side, without entering into the sample. The other bright fringes remain stationary as this fringe moves, because they arise from a double reflection, as discussed in parts (a) and (c). The single-reflection fringe can be used as an alignment indicator: at perpendicular illumination, it should fall back on the laser. This situation can only be extrapolated and not directly observed, because near the center the single-reflection fringe blends with the other bright fringes.

The reflection pattern has an advantage for measurements – the central fringes around $m = 0$ are very bright and always visible, so there is no problem with counting the fringes consecutively, even in bright ambient illumination. On the other hand, it has some disadvantages – the overall pattern is narrower, while the individual fringes are broader. The reflection setup is also more difficult to align: to observe a high-quality pattern, the laser beam must pass through the hole in the screen without scattering off the sides, and then it must reach the sample. On the other hand, this provides better feedback: if a clean pattern is obtained, it means that the alignment is good. This reduces the eventual error.

In all three configurations, we need an indicator for the point $x = 0$, from which both x and m should be counted. In Configuration 1 (reflection), given proper alignment, this point is given by the source of the laser beam, at the center of the hole in the screen. In Configurations 2 and 3 (transmission), it can be found by illuminating the screen directly without the sample, and recording the beam's location. Configuration 2 allows another method – to define $x = 0$ as

the midpoint between the two bright side patterns. See the solution to (k) for remarks on identifying the centers of these patterns.

When choosing the measurement points, one should choose the largest possible range for which the fringes can still be reliably counted. This will reduce the relative error in distance measurements. There is no need to record all the dozens of fringes in this range – a sample of 8 more-or-less evenly spaced points is enough.

In the white light measurements, it was clear that one must choose the largest possible distance y to minimize the relative errors. With the laser light, a compromise must be struck between this consideration and the widening of the laser beam with distance.

In Table 1, we present sample measurements from a reflection pattern at a distance $y = 652\text{mm} \pm 1\text{mm}$. The error in x can be understood in two different ways. First, one may consider the error due to the width of each separate fringe. Since we are looking for the center of the fringes, this error is smaller than the fringe width itself. It may estimated for transmitted light as $\Delta x = 0.5\text{mm}$ (same as the measurement resolution), and for reflected light as $\Delta x = 1\text{mm}$. On the other hand, one may consider the collective offset of all the fringes relative to the true $x = 0$ point. This can be estimated as half the fringe spacing, which may as large as $\Delta x = 1\text{cm}$. All the above error estimations should receive full credit. The corresponding errors in θ, calculated as in eq. (1), range from $\Delta\theta = 0.05^\text{o}$ to $\Delta\theta = 1^\text{o}$.

m	x (mm)	θ	$\sin\theta$
-29	-262.5	-21.93°	-0.3735
-20	-176	-15.11°	-0.2606
-10	-86.5	-7.56°	-0.1315
-1	-9	-0.79°	-0.0138
1	9	0.79°	0.0138
6	51	4.47°	0.0780
11	95.5	8.33°	0.1449
19	170	14.61°	0.2523
25	230.5	19.47°	0.3333

Table 1: Sample measurement results for part (i) from a reflection pattern at $y = 652\text{mm} \pm 1\text{mm}$

j. The student should use the relation $d\sin\theta = m\lambda$ and draw a linear graph of $\sin\theta$ as a function of m. The slope of this graph will be λ/d. The values of $\sin\theta$ for our sample experiment are shown in Table 1. Deciding on the appropriate error bars is problematic. First, we have the wide range of possible error estimations for θ from part (i). Second, it's difficult to estimate the relevance of the fringes' collective offset for the graph's slope. Therefore, error bars will not be graded.

The plot for our sample experiment is presented in Graph 1. Its slope reads:

$$k = 0.0130 \pm 0.0002\ (1.5\%)$$

This leads to the value $d = \lambda/k = 50.2\mu\text{m}$ for the spacing. The relative error in k is dominant over the relative error in λ, so we have $\Delta d = 0.8\mu\text{m}\ (1.5\%)$.

A student who doesn't take into account the doubled spacing between some of the fringes will get a less linear graph. A student who consistently uses the double spacing will get a linear graph, but his result for d will be too small by a factor of 2.

k. The student must use the bright patterns in the transmitted light (Configuration 2 or a rotated Configuration 3) to measure the deflection angle δ_0 or δ_{min}. Instead of looking through the sample, the student must use the screen. The peculiarities of the alignment process were already discussed in part (i). If part (i) was carried out with transmitted light, one may keep the same alignment and distance y to the screen.

The precise location corresponding to the desired deflection angle can be difficult to identify on the screen. The bright patches are several centimeters wide, and their center doesn't necessarily lie on one of the interference fringes. However, there is an oval-shaped aura defining these patches, and the center can be identified as the point where this aura is broadest. Alternatively, one can try to move the interference fringes by slight rotations of the sample, and use them to probe the bright patch; for instance, its center can be estimated as the place where the brightest possible fringe is obtained.

As for white light, the more precise measurement is obtained for the minimal deflection angle δ_{min}. Unlike with the white light, a larger distance y to the screen is not necessarily better, due to the broadening of the laser beam. In our sample experiment, we chose a moderate distance $y \approx 50\text{cm}$. Then instead of using the distance x between the un-deflected beam and the beam at minimal deflection, we used the distance $2x$ between the minimal deflections to the right and to the left. These two points were obtained by rotating the sample without changing any other element of the alignment. As a check, we in fact measured the distances x_1 and x_2 of the two points from the un-deflected beam, to see that they come out similar. The difference $(x_1 - x_2)/2$ can then serve as an error estimate for x.

The measurements from the sample experiment read:

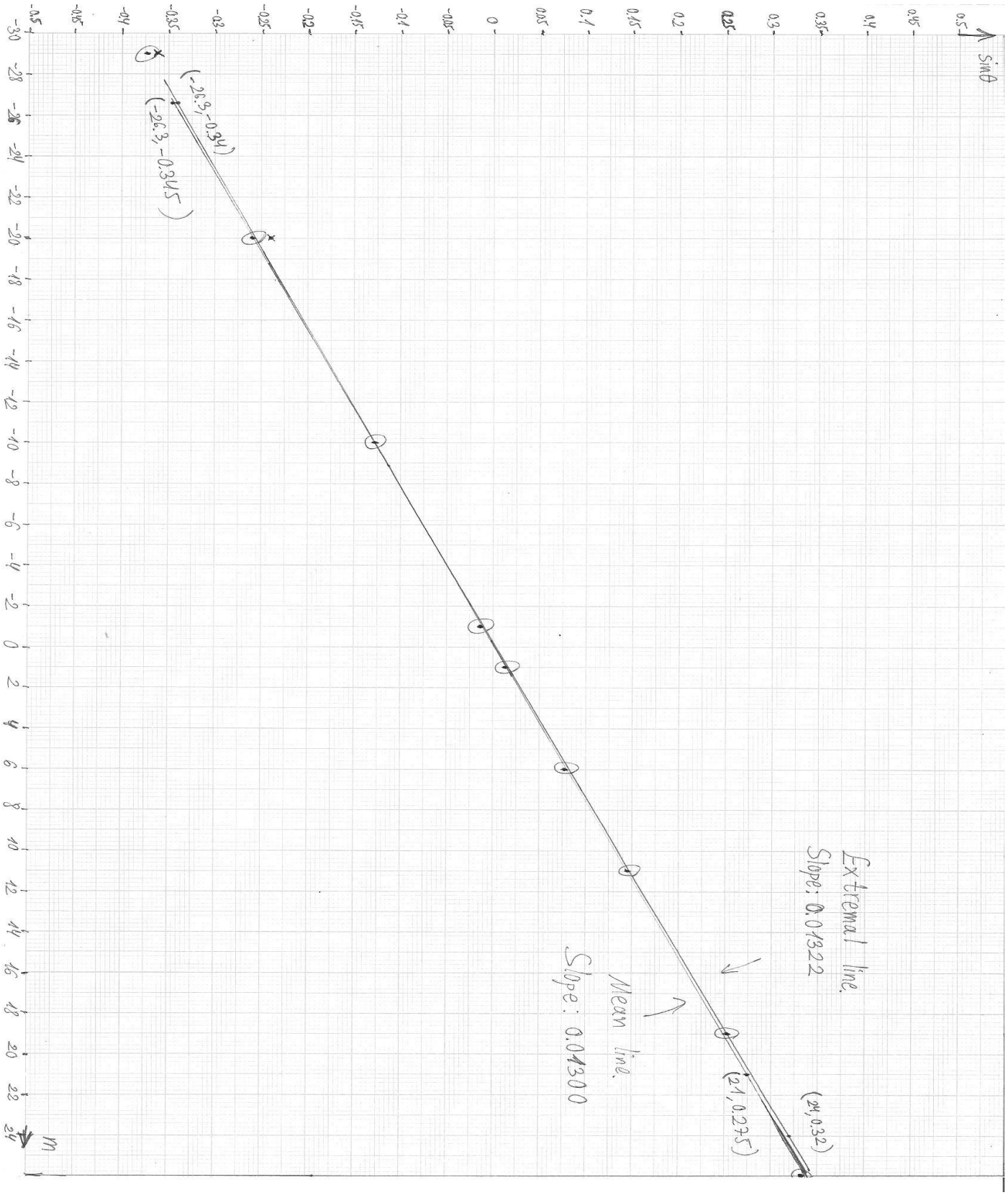
$$y = 495\text{mm} \pm 1\text{mm}; \quad x_1 = 276.5\text{mm}; \quad x_2 = 279\text{mm}$$

From x_1 and x_2, we derive $x = (x_1 + x_2)/2 = 277.7\text{mm}$, with an error $\Delta x = (x_1 - x_2)/2 = 1.5\text{mm}$. For δ_{min}, we get:

$$\delta_{min} = \tan^{-1}(x/y) = 29.3^{\text{o}}$$

Calculating the error as in eq. (1), we get $\Delta\delta_{min} = 0.002\text{rad} = 0.1^{\text{o}}$. This error is consistent with the scatter of several sample measurements.

Using eqs. (3) and (4) for the refractive index and its error, we get $n_r = 1.578 \pm 0.002$.



Graph 1: $\sin\theta$ as a function of m, with a linear trend line.